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Exact solutions for a scalar-tensor theory through symmetries

José Antonio Belinchón*

Departamento De Física, Facultad Ciencias Naturales
Universidad de Atacama, Copayapu 485, Copiapó, Chile
jose.belinchon@uda.cl

Pedro Dávila

Departamento De Estructuras y Física de Edificación, E.T.S.A.M.
Technical University of Madrid, 28040 Madrid, Spain
pedro.davila@upm.es

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In this paper, we study how to determine the unknown functions for the scalar tensor model $f(R, \phi)$ where the Ricci scalar is allowed to appear in a nonlinear way. The methods followed to determine these functions are: the matter collineation approach, the Lie group method and the Lagrangian collineation approach. We find several exact analytical solutions for a cosmological model with a FRW metric. We determine that some of the results are also valid for some anisotropic metric (e.g. the self-similar ones).

Keywords: Scalar tensor theories; exact solution; symmetries.

Mathematics Subject Classification 2010: 83D05, 83F05

1. Introduction

The physical and mathematical importance of the modified gravitational models has been recently pointed out by several authors (see for instance [26, 10, 1] and [14]), since this kind of theories explain in a better way the dynamics of the very early universe as well as its current acceleration. Obviously, although such class of theories is more general than the usual Jordan–Brans–Dicke models, they may be generalized in order to incorporate corrections to the Ricci scalar term as the $f(R)$ models (see for example [6]). Therefore, the purpose of this paper is to study the generalized $f(R, \phi)$ theories ([38, 17, 25, 2]), where, for example, as subclasses result, the $f(R)$ models (with $\phi = 0$) and the generalized scalar tensor theories with $f = F(\phi)R$ are studied in [3]. In particular, we are interested in studying the form that the different quantities may take in order that the field equations (FE) may be integrated.

*Corresponding author.

Therefore, it would be necessary to have a fundamental method according to which the form (or forms) of the potential as well as the other physical quantities could be fixed, and if it is possible, to calculate exact solutions to the proposed models. We have several geometric methods, such as the matter collineation (self-similar solutions), Lie groups and the Lagrangian collineation approach.

The study of self-similar (SS) models is quite important since, as it has been pointed out by Rosquist and Jantzen [33], they correspond to equilibrium points, and therefore a large class of orthogonal spatially homogeneous models is asymptotically SS at the initial singularity and approximated by exact perfect fluid (PF) or vacuum SS power law models. Exact SS power-law models can also approximate general Bianchi models at intermediate stages of their evolution [11]. From the geometrical point of view, self-similarity is defined by the existence of a homothetic vector field $\mathcal{H}$ in the spacetime, which satisfies the equation $L_{\mathcal{H}}g_{\mu\nu} = 2\alpha g_{\mu\nu}$ [8]. The geometry and physics at different points on an integral curve of a homothetic vector field (HVF) differ only by a change in the overall length scale, and in particular, any dimensionless scalar will be constant along the integral curves.

The existence of SS solutions (which implies that the scale factor follows a power-law solution) is just a manifestation of scaling symmetries. It is opportune to point out that scaling is not the most general form of symmetry. Symmetry methods are arguably the most systematic way of dealing with exact solutions of differential equations (partial as well as ordinary). In recent years, they have been successfully applied to various fields, such as gas dynamics, fluid mechanics, general relativity, etc. Among symmetries of a differential equation, those forming a one-parameter group of transformations can be determined algorithmically through the so-called *Lie algorithm*. Quite often, as in the $f(R,\phi)$ cosmological models, the FE of the model contain arbitrary functions whose functional forms cannot be fixed by any known laws. Since having symmetries is just a generic property, i.e. all equations do not admit symmetries, then symmetries can be used to determine such functions. This is known in the literature as *group modeling* [27]. The advantage of using such technique is that it is systematic. Therefore, by studying the forms of the unknown functions for which the FE admit symmetries, it is possible to uncover new integrable models.

Another method for determining the physical quantities is the use of the Lagrangian collineation approach (Noether-like symmetries). The idea of using Noether symmetries as a cosmological tool is not new in this kind of studies, for example, in [32], the authors proposed that the Noether point symmetry approach can be used as a selection rule for determining the form of the potential, that is, they take into account the geometry of the FE as a selection criterion, in order to fix the form of the potential. There exists a massive set of works on symmetries in scalar tensor theories so we can only cite a few of them. In [15], the author studied Noether symmetry of the hyperextended scalar–tensor theory for the FLRW models. In [28], the authors studied Scalar-Tensor cosmological models through Noether symmetries since the presence of symmetries implies that the dynamical system becomes integrable and then they can compute cosmological analytical solutions for specific functional forms of coupling and potential functions selected by the Noether Approach. Other recents works are [34, 29, 24]. Dynamically speaking, Noether symmetries are considered to play a central role in physical problems because they provide first integrals which can be utilized in order to simplify a given system of differential equations and thus to determine the integrability of the system. There are several approaches to study these symmetries; the geometrical one (see for instance [7] and the references therein), the dynamical Noether symmetry approach based on the Lie group method ([23] and [22]), and the developed in [37, 9]. In this paper, we shall follow the method proposed by Capozziello *et al.* in [7].

Therefore, the aim of this paper is to study $f(R, \phi)$ cosmological models by using several symmetry methods in order to determine the form of the physical quantities as for example, the potential as well as the other unknown functions that appear in the FE. In particular, we are interested in studying whether SS solutions exist and how each physical quantity must behave in order that the FE admits such class of solutions. In the same way, we formalize the use of power-law solutions (less restrictive than the SS ones) by studying the wave equation for the scalar field through the Lie group method. We also show how to use this approach in order to generate more solutions. Furthermore, we study the existence of Noether, like symmetries in order to find exact solutions in the framework of the flat FRW geometry.

The paper is organized as follows. In Sec. 2, we introduce the model and outline the FE for the model. In Sec. 3, we determine the exact form that each physical quantity may take in order that the FE admits exact SS solutions through the matter collineation approach. In Sec. 4, we study the wave equation for the scalar field through the Lie group method. We show how to generate several solutions by using this approach. Section 5 is devoted to studying the model through the Lagrangian collineation approach. We end up in Sec. 6 with a brief conclusion and discussion.

2. The Model

We consider a class of scalar-tensor theories of gravity represented by the action [30]

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}f(\phi, R) - \frac{1}{2}Z(\phi)\phi^{;\mu}\phi_{;\mu} - U(\phi) + \mathcal{L}_m\right], \tag{1}$$

where R is the Ricci scalar and $\mathcal{L}_m$ is a classical matter Lagrangian including also minimally coupled scalar fields, if any. We disregard any possible coupling of our scalar field with ordinary matter, radiation and dark matter [12, 31].

We assume a standard Friedman–Robertson–Walker (FRW) form for the unperturbed background metric and we restrict ourselves to a spatially flat universe, that is,

$$ds^2 = -dt^2 + a^2(t)\sum_{i=1}^{3}(dx^i)^2, \tag{2}$$

so we are using the signature $(-,+,+,+)$ and where an expression that will be useful in the following is that of the Ricci scalar,

$$R = 6(\dot{H} + 2H^2), \quad H = \dot{a}/a. \tag{3}$$

We are using units where $8\pi G = c \equiv 1$, and we will choose the relation $G_{\mu\nu} = T_{\mu\nu}$ to identify $T_{\mu\nu}$. Here $G_{\mu\nu}$ is the Einstein tensor, and all the other contributions have been absorbed in $T_{\mu\nu}$; as noted in ([19–21]), and therefore, if one writes the gravitational field equation in this form, then $T_{\mu\nu}$ can be treated as an effective stress–energy tensor, which allows to use the standard Einstein equations by simply replacing the fluid quantities with the effective ones. By defining $F \equiv \partial f/\partial R$, the gravitational FE derived by the action (1) are

$$FG_{\mu\nu} = T^m_{\mu\nu} + Z\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\sigma}\phi^{,\sigma}\right) + F_{,\mu;\nu} - g_{\mu\nu}\Box F - Ug_{\mu\nu} + \frac{1}{2}(f - FR)g_{\mu\nu}, \tag{4}$$

and the wave equation (see [20])

$$2Z\Box\phi + Z_{,\phi}\phi^{,\sigma}\phi_{,\sigma} + f_{,\phi} - 2U_{,\phi} = 0. \tag{5}$$

In this paper, we consider that the matter content is described by a PF whose energy–momentum tensor is defined by

$$T^m_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \tag{6}$$

where ρ is the energy density of the fluid, p the pressure and they are related by the equation of state $p = \gamma\rho$, $(\gamma \in (-1,1])$, and $u^\mu = (1,0,0,0)$ is the 4-velocity.

Equations for the background are

$$3FH^2 = \rho + \frac{Z}{2}\dot{\phi}^2 - 3H\dot{F} + \frac{1}{2}(RF - f) + U, \tag{7}$$

$$F(2\dot{H} + 3H^2) = -p - \frac{Z}{2}\dot{\phi}^2 - 2H\dot{F} - \ddot{F} + \frac{1}{2}(RF - f) + U, \tag{8}$$

$$2Z(\ddot{\phi} + 3H\dot{\phi}) = f_\phi - Z_\phi\dot{\phi}^2 - 2U_\phi, \tag{9}$$

furthermore, the continuity equations for the individual fluid components are not directly affected by the changes in the gravitational field equation, and for the i-th component

$$\dot{\rho}_i + 3H(\rho_i + p_i) = 0. \tag{10}$$

2.1. *Simplifying assumptions*

We may suppose that the function $f(\phi, R)$ can be split in the following way

$$f(\phi, R) = h(R)g(\phi). \tag{11}$$

3. Matter Collineation Approach

Our purpose will be to determine the exact form that must be followed by each physical quantity in order that the FE admit SS solutions. We shall use the tactic of the matter collineations approach, which guarantees us the existence of SS solutions following the method developed in a previous paper (see [3]).

Self-similarity is defined by the existence of a HVF V in the spacetime ([5, 13]), which satisfies

$$L_V g_{\mu\nu} = 2\alpha g_{\mu\nu}, \tag{12}$$

where $g_{\mu\nu}$ is the metric tensor, L_V denotes Lie differentiation along the vector field $V \in \mathfrak{X}(M)$ and α is a constant (see for general reviews [8, 16]). If we consider the Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, where $T_{\mu\nu}$ is an effective stress–energy tensor, then if the spacetime is homothetic, the energy–momentum tensor of the matter fields must satisfy $L_V T_{\mu\nu} = 0$. Nevertheless, in this work, we are not interested in finding the set of vector fields $V \in \mathfrak{X}(M)$, that verify such equation, otherwise, knowing that the HVF $\mathcal{H}$ (see for example [8]), that is, $L_{\mathcal{H}} g_{\mu\nu} = 2g_{\mu\nu}$, then $\mathcal{H}$ is also a matter collineation, i.e. $L_{\mathcal{H}} T_{\mu\nu} = 0$. We use this fact to determine the behavior of the main physical quantities in order that the FE admit SS solutions (see [16]).

Therefore, we calculate

$$L_{\mathcal{H}} T^{\mathrm{eff}}_{\mu\nu} = 0, \tag{13}$$

where $\mathcal{H}$ is a HVF i.e. it verifies the equation: $L_{\mathcal{H}} g_{\mu\nu} = 2g_{\mu\nu}$, for some metric and where $T^{\mathrm{eff}}_{\mu\nu}$ is the effective stress–energy tensor. For this purpose, we have shown in [3] that it is enough to calculate $L^{(i)}_{\mathcal{H}} T_{\mu\nu} = 0$, for each component of the stress–energy tensor. We are considering a FRW metric, thus the HVF yields (see for instance [18])

$$\mathcal{H} = t\partial_t + (1 - a_1)(x\partial_x + y\partial_y + z\partial_z), \tag{14}$$

where $a_1 \in \mathbb{R}$, is a numerical constant, note that the scale factor must behaves as, $a(t) = t^{a_1}$, $a_1 \in \mathbb{R}^+$. We may do such simplification because, as we have shown in [3], all the physical quantities are homogeneous, that is, they only depend on time t, then, the unique equation of $L^{(i)}_{\mathcal{H}} T_{\mu\nu} = 0$, that is interesting for us is the one corresponding to the temporal coordinate $t\partial_t$.

We start by defining $T^{\mathrm{eff}}_{\mu\nu}$ as follows:

$$T^{\mathrm{eff}}_{\mu\nu} = {}^M T_{\mu\nu} + {}^Z T_{\mu\nu} + {}^\phi T_{\mu\nu} + {}^U T_{\mu\nu} + {}^R T_{\mu\nu}, \tag{15}$$

where

$${}^M T_{\mu\nu} = \frac{T_{\mu\nu}}{F} = \frac{1}{F}((\rho + p)u_\mu u_\nu + p g_{\mu\nu}), \tag{16}$$

$${}^Z T_{\mu\nu} = \frac{Z}{F}\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2} g_{\mu\nu}\phi_{,\sigma}\phi^{,\sigma}\right), \tag{17}$$

$$ {}^{\phi}T_{\mu\nu} = \frac{1}{F}(F_{,\mu;\nu} - g_{\mu\nu}\Box F), \tag{18} $$

$$ {}^{U}T_{\mu\nu} = -\frac{U}{F}g_{\mu\nu}, \tag{19} $$

$$ {}^{R}T_{\mu\nu} = \frac{1}{2F}(f - FR)g_{\mu\nu} = \frac{1}{2h_R g}(hg - h_R g R)g_{\mu\nu} = \frac{1}{2}(hh_R^{-1} - R)g_{\mu\nu}, \tag{20} $$

since (it does not depend on ϕ) note that $f = hg$, and $F = h_R g$.

Then $L_{\mathcal{H}}({}^{M}T) = 0$ yields

$$ -\frac{g_\phi \dot{\phi}}{g} - \frac{h_{RR}\dot{R}}{h_R} + \frac{\dot{\rho}}{\rho} = -2t^{-1}, \tag{21} $$

with $\dot{\phi} = d\phi/dt$, that is, a dot means derivative with respect to t, then integrating it, we find the relationship between the quantities

$$ \rho(h_R g)^{-1} = t^{-2}. \tag{22} $$

In the same way, from $L_{\mathcal{H}}({}^{Z}T) = 0$, we get

$$ \frac{Z_\phi \dot{\phi}}{Z} - \frac{g_\phi \dot{\phi}}{g} - \frac{h_{RR}\dot{R}}{h_R} + 2\frac{\ddot{\phi}}{\dot{\phi}} = -2t^{-1}, \tag{23} $$

then

$$ Z\dot{\phi}^2(h_R g)^{-1} = t^{-2}. \tag{24} $$

Now ${}^{U}T_{\mu\nu} = Ug_{\mu\nu}/F$, by performing the same calculations, we conclude that

$$ \frac{U_\phi \dot{\phi}}{U} - \frac{g_\phi \dot{\phi}}{g} - \frac{h_{RR}\dot{R}}{h_R} = -2t^{-1}, \quad U(h_R g)^{-1} = t^{-2}. \tag{25} $$

With regard to the component ${}^{R}T_{\mu\nu}$: $L_{\mathcal{H}}({}^{R}T_{\mu\nu}) = L_{\mathcal{H}}(\frac{1}{2}(hh_R^{-1} - R)g_{\mu\nu}) = 0$. The first term is: $L_{\mathcal{H}}(T_{00}^{f}) = 0$, that we may write in the following form

$$ -th_{RR}h_R^{-1}\dot{R}h = 2h_R R - 2h, \tag{26} $$

and taking into account the fact that $t\dot{R} = -2R$ (in the framework of SS solutions and for the FRW metric), then the equation yields

$$ h_{RR} = \frac{h_R^2}{h} - \frac{h_R}{R}, \tag{27} $$

whose solution is

$$ h = C_1 R^r, \quad r \in \mathbb{R}, \tag{28} $$

where C_1 is an integrating constant. With the other components, $L_{\mathcal{H}}({}^{R}T_{1i}) = 0$, we only obtain restriction on the scale factors, but we already know how it behaves from the HVF. Therefore, we have obtained the following result

$$ h = C_1 R^r \quad r \in \mathbb{R}, \quad R = R_0 t^{-2}, \tag{29} $$

for the FRW metric $R_0 = 6a_1(2a_1 - 1)$, since $a(t) = t^{a_1}$, $a_1 \in \mathbb{R}^+$.

To end, and taking into account the previous results, we go next to calculate $L_{\mathcal{H}}({}^{\phi}T_{\mu\nu}) = 0$, where ${}^{\phi}T_{\mu\nu} = (F_{,\mu;\nu} - g_{\mu\nu}\Box F)/F$. By calculating the first component of this equation, we get

$$\frac{g_{\phi\phi}\dot{\phi}}{g_{\phi}} - \frac{g_{\phi}\dot{\phi}}{g} + \frac{\ddot{\phi}}{\dot{\phi}} + \frac{1}{t} = 0. \tag{30}$$

This ODE admits several solutions. For example, we may assume that g follows a power law solution, $g = g_0\phi^n$, then

$$g = g_0\phi^n, \quad \phi = \phi_0 t^m, \quad n, m \in \mathbb{R}. \tag{31}$$

In the second place, by solving the above ODE for g and ϕ, we obtain

$$g = g_0 e^{C_1\phi}, \quad \phi = \phi_0 \ln t, \quad g_0, \phi_0, C_1 \in \mathbb{R}, \tag{32}$$

both solutions, Eqs. (31) and (32) are compatible with the power-law solution for the scale factor.

Now, if we consider the expression from the matter conservation: $\rho = \rho_0 a^{-3(\gamma+1)} = \rho_0 t^{-3a_1(\gamma+1)}$, and taking into account that $\rho g^{-1} \approx t^{-2r}$, then we get the following relationships. For the solution (31) $g = t^l$, with $l = mn$, then $l = 2r - 3a_1(\gamma + 1)$, and therefore

$$\begin{aligned}
\rho(h_R g)^{-1} &= t^{-2}, \quad \rho \approx t^{B}, \quad B = -3a_1(\gamma+1),\\
Z\dot{\phi}^2(h_R g)^{-1} &= t^{-2}, \quad Z \approx t^{A}, \quad Z \approx \phi^{A/m}, \quad A = 2(1-m) + B,\\
U(h_R g)^{-1} &= t^{-2}, \quad U \approx t^{B}, \quad U \approx \phi^{B/m},
\end{aligned} \tag{33}$$

with the restriction $l - 2r < 0$, in order to obtain an energy density and a potential decreasing on time. For the solution (32) $g = g_0 t^{C_1\phi_0}$, then $C_1\phi_0 = 2r - 3a_1(\gamma+1)$, and therefore,

$$\begin{aligned}
\rho(h_R g)^{-1} &= t^{-2}, \quad \rho \approx t^{-A}, \quad A = 3a_1(\gamma+1),\\
Z\dot{\phi}^2(h_R g)^{-1} &= t^{-2}, \quad Z \approx t^{2-A}, \quad Z \approx e^{(2-A)\phi},\\
U(h_R g)^{-1} &= t^{-2}, \quad U \approx t^{-A}, \quad U \approx e^{-A\phi},
\end{aligned} \tag{34}$$

with the restriction $3a_1(\gamma + 1) = A > 0$, in order to obtain an energy density and a potential decreasing on time.

Remark 1. Relationships from Dimensional Analysis (DA). From the FE (7)

$$3H^2 = \frac{\rho}{F} + \frac{Z}{2F}\dot{\phi}^2 - 3H\frac{\dot{F}}{F} + \frac{1}{2F}(RF - f) + \frac{U}{F},$$

with $F = h_R g$, and $\dot{F} = h_{RR}\dot{R}g + h_R\dot{\phi}g_{\phi}$, we note that $[H^2] = T^{-2}$, then we have

$$[H^2] = T^{-2} = [\rho F^{-1}] = [ZF^{-1}\dot{\phi}^2] = [H\dot{F}F^{-1}] = [(RF - f)F^{-1}] = [UF^{-1}],$$

so developing the brackets, we get

$$\rho(h_R g)^{-1} = t^{-2}, \quad Z\dot{\phi}^2(h_R g)^{-1} = t^{-2}, \quad U(h_R g)^{-1} = t^{-2},$$

that is, we have obtained the same relationships (as it is expected) but in a trivial way. Nevertheless, following this procedure, we are not able to determine the form of each function as in Eqs. (33) and (34).

4. Lie Groups Approach

We have proved how each physical quantity must behave under the hypothesis of self-similarity. In this section, we shall follow another approach, which allows us to find the same and more solutions. We study through the Lie group method the wave equation for the scalar function

$$2Z(\ddot{\phi}+\theta\dot{\phi}) = f_\phi - Z_\phi\dot{\phi}^2 - 2U_\phi, \tag{35}$$

that is, Eq. (35) is of the general form $\ddot{\phi} = \psi(t,\phi,\dot{\phi})$.

Roughly speaking, a symmetry, $X = \xi(t,\phi)\partial_t + \eta(t,\phi)\partial_\phi$, of a differential equation is an invertible transformation that leaves it form-invariant. By applying the standard Lie procedure (see for instance [4, 36, 22]), we need to solve the following overdetermined system of linear partial differential equations for η and ξ (from the extended infinitesimal or prolonged transformations), which allows us to determine the set of the symmetries admitted by Eq. (35). A vector field X, $X = \xi(t,\phi)\partial_t + \eta(t,\phi)\partial_\phi$, is a symmetry of (35) if

$$\begin{aligned} -\xi\psi_t - \eta\psi_\phi + \eta_{tt} + (2\eta_{t\phi} - \xi_{tt})\dot{\phi} + (\eta_{\phi\phi} - 2\xi_{t\phi})\dot{\phi}^2 - \xi_{\phi\phi}\dot{\phi}^3 \\ + (\eta_\phi - 2\xi_t - 3\dot{\phi}\xi_\phi)\psi - [\eta_t + (\eta_\phi - \xi_t)\dot{\phi} - \dot{\phi}^2\xi_\phi]\psi_{\dot{\phi}} = 0. \end{aligned} \tag{36}$$

Thus, our approach consists in imposing a particular symmetry and to deduce the exact form that acquires the unknown functions, that is, ϕ, f, Z and U, by solving the system of PDE (36). The imposed symmetry induces a change of variables which usually reduces Eq. (35) to an integrable ODE. However, sometimes, it is not possible to find a solution of such ODE. For this reason, the knowledge of one symmetry X might suggests the form of a particular solution as an invariant of the operator X, i.e. a solution of $dt/\xi(t,\phi) = d\phi/\eta(t,\phi)$. This particular solution is known as an invariant solution (generalization of similarity solution).

Therefore, we study Eq. (35), and rewrite it as follows:

$$\ddot{\phi} = -3H\dot{\phi} + \frac{f_\phi}{2Z} - \frac{Z_\phi}{2Z}\dot{\phi}^2 - \frac{U_\phi}{Z}, \tag{37}$$

we use the notation $\dot{\phi} = d\phi/dt$, $U_\phi = dU/d\phi$, etc.

By studying Eq. (37) under the assumption $f(\phi,R) = h(R)g(\phi) = h(t)g(\phi)$, we find (following the standard Lie procedure) the system of PDE

$$Z_\phi\xi_\phi - 2Z\xi_{\phi\phi} = 0, \tag{38}$$

$$(ZZ_{\phi\phi} - Z_\phi^2)\eta + ZZ_\phi\eta_\phi + 12Z^2H\xi_\phi + 2Z^2\eta_{\phi\phi} - 4Z^2\xi_{t\phi} = 0, \tag{39}$$

$$3(2U_\phi - h(t)g_\phi)\xi_\phi + 6Z(H\xi_t + \xi H_t) + 4Z\eta_{t\phi} - 2Z\xi_{tt} + 2Z_\phi\eta_t = 0, \tag{40}$$

$$\begin{aligned}&[h(Z_\phi g_\phi - Zg_{\phi\phi}) + 2(ZU_{\phi\phi} - Z_\phi U_\phi)]\eta + Z(hg_\phi - 2U_\phi)\eta_\phi \\ &\quad + 2Z(2U_\phi - hg_\phi)\xi_t - h'Zg_\phi\xi + 2Z^2(3H\eta_t + \eta_{tt}) = 0.\end{aligned} \tag{41}$$

Thus, by imposing a symmetry $[\xi, \eta]$ in the above system of PDE, we shall be able to determine the form of the unknown functions, Z, H, h and g in order to obtain an integrable model.

4.1. *Symmetry* $[nt, \phi]$

By considering the symmetry $[\xi = nt, \eta = \phi]$, then, from Eq. (39), we obtain

$$(ZZ_{\phi\phi} - Z_\phi^2)\phi + ZZ_\phi = 0, \quad \Rightarrow \quad Z = Z_0\phi^m, \quad m \in \mathbb{R}. \tag{42}$$

Now, from Eq. (40), we get

$$H + tH' = 0, \quad \Rightarrow \quad H = a_1 t^{-1}, \quad a_1 \in \mathbb{R}^+, \tag{43}$$

this implies that the scale factor behaves as: $a = t^{a_1}$, and therefore $R = R_0 t^{-2}$.

From Eq. (41), splitting in g and U, and simplifying we get

$$[(U_{\phi\phi} - m\phi^{-1}U_\phi)]\phi - U_\phi + 2nU_\phi = 0, \tag{44}$$

$$h(m\phi^{-1}g_\phi - g_{\phi\phi})\phi + hg_\phi - 2hg_\phi n - h'g_\phi nt = 0, \tag{45}$$

so, we find

$$U_{\phi\phi} = AU_\phi\phi^{-1}, \quad \Rightarrow \quad U = C_1 + C_2\phi^{A+1}, \tag{46}$$

with $A = 1 + m - 2n$, and $C_1, C_2 \in \mathbb{R}$, while

$$g_{\phi\phi} = B\phi^{-1}g_\phi \quad \Rightarrow \quad g = C_3 + C_4\phi^{B+1}, \tag{47}$$

where $B = 1 + m - 2n - h'nt/h$, and $C_3, C_4 \in \mathbb{R}$, that is, they are constant of integration.

Therefore, we have obtained the following solution

$$\begin{aligned} f(\phi, R) = h(R)g(\phi) &= h(R)(C_3 + C_4\phi^{B+1}), \\ U &= C_1 + C_2\phi^{A+1}, \quad A = 1 + m - 2n, \\ Z &= Z_0\phi^m, \quad m \in \mathbb{R}, \\ H &= a_1 t^{-1}, \quad a_1 \in \mathbb{R}, \quad a = t^{a_1}, \end{aligned} \tag{48}$$

with $B = 1 + m - 2n - h'nt/h$, thus (without lost of generality)

$$\begin{aligned} f(\phi, R) = h(R)C_4\phi^{B+1}, \quad U = U_0\phi^{A+1}, \\ Z = Z_0\phi^m, \quad H = a_1 t^{-1}. \end{aligned} \tag{49}$$

With regard to the function $h(R)$, we set $h'nt/h = \text{const}$, since all the quantities in B are numerical constants, therefore

$$\frac{h'}{h}nt = c \;\; \Rightarrow \;\; h = C_5 t^{\frac{c}{n}}, \tag{50}$$

where $c \in \mathbb{R}$, such that $c \neq -2n$, and $nc \in \mathbb{R}^-$, this result implies that

$$h(R) = C_6 R^r = R_0^r t^{-2r} = R_0^r t^{c/n}, \quad r = -c/2n. \tag{51}$$

In this way

$$\begin{aligned} f(\phi, R) = h(R)g(\phi) &= f_0 \phi^{B+1} R^r, \\ U &= U_0 \phi^{A+1}, \\ Z &= Z_0 \phi^m, \quad m \in \mathbb{R}, \\ H &= a_1 t^{-1}, \quad a = t^{a_1}, \\ R &= 6a_1(2a_1 - 1)t^{-2}, \end{aligned} \tag{52}$$

with $B = 1 + m - 2n - c$, $A = 1 + m - 2n$, $B = A - c$, $-2rn = c$, $a_1 \in \mathbb{R}$, so $R^r = R_0^r t^{-2r} = R_0^r t^{c/n}$.

Action (1) collapses to

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} \phi^{2+m+2n(r-1)} R^r - \frac{1}{2} \phi^m \phi^{;\mu} \phi_{;\mu} - \phi^{2+m-2n} + \mathcal{L}_m \right]. \tag{53}$$

4.1.1. *Wave equation*

We now try to find a solution for the scalar field ϕ by studying the wave equation (37) introducing the obtained results (52) into it, then we obtain

$$\begin{aligned} \ddot{\phi} = &-\frac{m}{2}\phi^{-1}\dot{\phi}^2 - 3a_1 t^{-1}\dot{\phi} + (2 + m - 2n - c)\frac{R_0^r}{2Z_0} t^{\frac{c}{n}} \phi^{1-2n-c} \\ &- (2 + m - 2n)\frac{U_0}{Z_0}\phi^{1-2n}, \end{aligned} \tag{54}$$

finding a particular solution, $\phi = \phi_0 t^{1/n}$, $\phi_0 \in \mathbb{R}$, which is compatible with power-law for the scale factor: $a = t^{a_1}$. This particular solution is also the invariant solution induced by the symmetry $[\xi = nt, \eta = \phi]$, that is,

$$\frac{dt}{\xi} = \frac{d\phi}{\eta} \Rightarrow \phi = \phi_0 t^{1/n}, \tag{55}$$

thus

$$\begin{aligned} h(R) = h(t) &= C_9 t^{\frac{c}{n}}, \\ g(\phi) &= C_1 \phi^{B+1} = g_0 t^{\frac{1}{n}(2+m-2n-c)}, \end{aligned}$$

$$
\begin{aligned}
f(\phi,R) = h(R)g(\phi) &= f_0 t^{\frac{1}{n}(2+m-2n)},\\
Z &= Z_0 t^{m/n},\\
U &= U_0 t^{\frac{1}{n}(2+m-2n)},\\
H &= a_1 t^{-1}, \quad a = t^{a_1},
\end{aligned}
\tag{56}
$$

with $c\in\mathbb{R}$, $c\neq -2n$, $nc\in\mathbb{R}^-$, and $B = 1+m-2n-c$. Furthermore, this solution verifies Eq. (33), so we may say that the soltion is SS.

As it is observed, from the first of the FE Eq. (7)

$$3FH^2 = \rho + \frac{Z}{2}\dot{\phi}^2 - 3H\dot{F} + \frac{1}{2}(RF-f) + U, \tag{57}$$

the invariant solution is consistent from the dimensional point of view, since all the quantities involved have the same dimensional equation (have the same order of magnitude $t^{\frac{1}{n}(2+m-2n)}$), that is, $[f] = [Z\dot{\phi}^2] = [U] = [\rho]$, where $\rho = \rho_0 a^{-3(\gamma+1)} = \rho_0 t^{-3a_1(\gamma+1)}$, and therefore $-3a_1(\gamma+1) = (2+m-2n)/n$, so

$$a_1 = \frac{2(n-1)-m}{3n(\gamma+1)}, \tag{58}$$

that is, $a_1 = a_1(n,m,\gamma)$, where we assume that $a_1 > 0$. The deceleration parameter, $q = d(H^{-1})/dt - 1$, is therefore

$$q = \frac{3n(\gamma+1)}{2(n-1)-m} - 1. \tag{59}$$

Now, we may find restrictions on the parameters (n,m,γ) under the assumptions $a_1>0$, $q<0$ and a U decreasing ($U\searrow$, it mimics a variable cosmological constant). For example, if we set, $m=-2$, we find that $a_1 = 2/3(\gamma+1)$, and therefore $q = (3\gamma+1)/2$ and $U = U_0 t^{-2}$, finding in this way that $q<0$, iff $\gamma < -1/3$.

In the same way, if we set $m=0$, induced gravity like model (IG), then the action yields

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}\phi^{2+2n(r-1)}R^r - \frac{1}{2}\phi^{;\mu}\phi_{;\mu} - \phi^{2(1-n)} + \mathcal{L}_m\right], \tag{60}$$

while the quantities behave as

$$
\begin{gathered}
\phi \approx t^{1/n}, \quad h(t) \approx t^{-2r}, \quad g(\phi) \approx t^{\frac{2(1+n(r-1))}{n}},\\
f(\phi,R) \approx t^{\frac{2}{n}(1-n)}, \quad Z \approx t^0, \quad U \approx t^{\frac{2}{n}(1-n)},
\end{gathered}
\tag{61}
$$

then we get $a_1>0$, $q<0$, and $U\searrow$, that is, $a_1 = 2(n-1)/3n(\gamma+1)$, $q = (3n(\gamma+1)/2(n-1)) - 1$ and $U_0 t^{2(1-n)/n}$, thus we find that $-1<\gamma$ and $n<0$.

4.2. *Symmetry* $[n, \phi]$

If we consider the symmetry $[\xi = n, \eta = \phi]$, then the system (38)–(41) yields

$$\left(Z_{\phi\phi} - \frac{Z_\phi^2}{Z}\right)\eta + Z_\phi = 0, \tag{62}$$

$$6Z\xi H' = 0, \tag{63}$$

$$\eta U_{\phi\phi} - \left(\eta_\phi - 2\xi_t + \frac{Z_\phi}{Z}\eta\right)U_\phi = 0, \tag{64}$$

$$\eta g_{\phi\phi} - \left(\frac{Z_\phi}{Z}\eta + \eta_\phi - 2\xi_t - \frac{h'}{h}\xi\right)g_\phi = 0, \tag{65}$$

so, from Eqs. (62)–(63), we get

$$Z = Z_0\phi^m, \quad m \in \mathbb{R}, \quad H = \text{const} = a_1, \quad a = \exp(a_1 t), \quad q = -1, \tag{66}$$

then, since $H = \text{const} = a_1$, this means that the scalar curvature R is also constant ($R = 12H^2$), thus

$$R = \text{const.} \quad \Rightarrow \quad h(R) = \text{const.} \quad \Rightarrow \quad h' = 0, \tag{67}$$

and therefore, Eqs. (64)–(65) yield

$$\phi U_{\phi\phi} - (1+m)U_\phi = 0, \quad U = C_1 + C_2\phi^{m+2}, \tag{68}$$

$$\phi g_{\phi\phi} - (1+m)g_\phi = 0, \quad g = C_3 + C_4\phi^{m+2}, \tag{69}$$

with $C_i \in \mathbb{R}$.

Therefore, we have obtained the following set of solutions

$$\begin{aligned}
Z &= Z_0\phi^m, \quad m \in \mathbb{R},\\
H &= \text{const} = a_1, \quad a = \exp(a_1 t), \quad q = -1,\\
R &= \text{const.}\\
g &= g_0\phi^{m+2},\\
h(R) &= \text{const.}, \quad h = R^r, \quad r \in \mathbb{R},\\
f &= hg = R^r\phi^{m+2},\\
U &= U_0\phi^{m+2},\\
\rho &= \rho_0 a^{-3(\gamma+1)} = \rho_0 e^{-3a_1(\gamma+1)t},
\end{aligned} \tag{70}$$

which are consistent, at least, from the dimensional point of view. In this way, action (1) collapses to

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R^r\phi^{m+2} - \frac{1}{2}Z_0\phi^m\phi^{;\mu}\phi_{;\mu} - U_0\phi^{m+2} + \mathcal{L}_m\right]. \tag{71}$$

4.2.1. *Wave equation*

We now try to solve the wave equation (37) for the scalar field, by introducing the above results, it yields

$$\ddot{\phi} = -A\phi^{-1}\dot{\phi}^2 - B\dot{\phi} + C\phi, \tag{72}$$

where $A = m/2$, $B = 3a_1$, and $C = (g_0(12a_1^2)^r - 2U_0)(m+2)/2Z_0$, finding in this way the following general solution:

$$\phi = \exp\left(-\frac{B}{A+1}t\right)(C_1\exp(M_+) - C_2\exp(M_-))^{\frac{1}{(A+1)}}, \tag{73}$$

with $C_1, C_2 \in \mathbb{R}$, and

$$M_\pm = \frac{1}{2}(B \pm \sqrt{B^2 + 4C(A+1)})t, \tag{74}$$

note that for this general solution $m \neq -2$, which is new in the literature to the best of our knowledge. We have performed a numerical analysis of this solution, and taking into account that the potential must be decreasing on time, then we may conclude that $C_1 = 0$. In this way, if $C_1 = 0$, then Eq. (73) yields

$$\phi = \phi_0 e^{-Kt}, \quad K = \frac{\sqrt{B^2 + 4C(A+1)}}{A+1}. \tag{75}$$

We also may find the particular (invariant) solution

$$\phi = \phi_0\exp(t/n), \tag{76}$$

where n takes the following values from Eq. (72)

$$an^2 + bn + c = 0, \tag{77}$$

being $a = (g_0(12a_1^2)^r - 2U_0)(m+2)/Z_0$, $b = -6a_1$, and $c = -(m+2)$, therefore

$$n = -\frac{c}{b}, \quad \text{if } b \neq 0 \wedge a = 0, \tag{78}$$

thus

$$n = -\frac{(m+2)}{6a_1} \quad \text{if}\left(\frac{g_0(12a_1^2)^r}{Z_0} - 2\frac{U_0}{Z_0}\right)(m+2) = 0, \tag{79}$$

that is $m = -2$, or $g_0(12a_1^2)^r = 2U_0$, but if $m = -2$, then $n = 0$ (since $c = 0$, so we have to ruled out this particular case, $m = -2$), and the other solution

$$n = \frac{3a_1 \pm \sqrt{9a_1^2 + \left(\frac{g_0(12a_1^2)^r}{Z_0} - 2\frac{U_0}{Z_0}\right)(m+2)^2}}{\left(\frac{g_0(12a_1^2)^r}{Z_0} - 2\frac{U_0}{Z_0}\right)(m+2)}, \tag{80}$$

with $(g_0(12a_1^2)^r - 2U_0)(m+2)/Z_0 \neq 0$, and $9a_1^2 > (g_0(12a_1^2)^r - 2U_0)(m+2)^2/Z_0$. Therefore, this solution is valid for any value of m except the case $m = -2$.

From the conservation equation $\rho = \rho_0 a^{-3(\gamma+1)}$ and the first of the FE Eq. (7), the invariant solution is consistent from the dimensional point of view, and therefore, $-3a_1(\gamma+1) = (2+m)/n$, so

$$a_1 = -\frac{m+2}{3n(\gamma+1)}, \tag{81}$$

noting that $a_1 > 0$ if $n < 0$. With regard to the potential, $U = U_0\phi^{m+2}$, it is decreasing only if $(m+2)/n < 0$, so $n < 0$ is a consistent value. In this case, we cannot obtain any restriction on the free parameters from the deceleration parameter q, since $q = -1$ (accelerating solution) for any value of n, m and γ. Usually m takes the values $m = -1$ (Jordan–Brans–Dicke like solution), $m = 0$ (induced gravity like model) and $m = -2$ (scalar like model).

4.3. *Symmetry* $[at, b]$

If we consider the symmetry $[at, b]$, then the system (38)–(41) collapses to

$$ZZ_{\phi\phi} - Z_\phi^2 = 0, \tag{82}$$

$$H + tH_t = 0, \tag{83}$$

$$h(Z_\phi g_\phi - Zg_{\phi\phi})b - h'Zg_\phi at - 2Zahg_\phi = 0, \tag{84}$$

$$2b(ZU_{\phi\phi} - Z_\phi U_\phi) + 4ZU_\phi a = 0, \tag{85}$$

thus

$$\begin{aligned}
&Z_{\phi\phi} = \frac{Z_\phi^2}{Z} \;\Rightarrow\; Z = Z_0 e^{C_1\phi},\\
&\frac{H_t}{H} = -\frac{1}{t} \;\Rightarrow\; H = \frac{a_1}{t} \;\Rightarrow\; a = t^{a_1}, \quad R = R_0 t^{-2},\\
&U_{\phi\phi} = \left(C_1 - \frac{2a}{b}\right)U_\phi \;\Rightarrow\; U = C_2 + U_0 e^{(C_1-2n)\phi}, \quad n = \frac{a}{b},
\end{aligned} \tag{86}$$

taking into account these results, then

$$g_{\phi\phi} = \left(C_1 - n\left(2 + t\frac{h'}{h}\right)\right)g_\phi, \tag{87}$$

and therefore

$$g = C_3 + \frac{1}{C_4}e^{C_4(C_5+\phi)}, \quad h = h_0 t^{\frac{1}{n}(C_1-C_4-2n)}. \tag{88}$$

If we set $C_2 = C_5 = 0$, $n = a/b$, then (setting $C_1 = m$, $C_4 = k$)

$$Z = Z_0 e^{m\phi}, \quad H = \frac{a_1}{t}, \quad U = U_0 e^{(m-2n)\phi}, \quad g = \frac{1}{k}e^{k\phi}, \quad h = h_0 t^{(m-k-2n)} = R^r, \tag{89}$$

while the invariant solution induced by the symmetry $[nt, 1]$ is

$$\phi = \frac{1}{n}\ln t, \tag{90}$$

noting that this solution is quite similar the obtained one in Eq. (32). Thus, the action yields

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R^r e^{C_4\phi} - \frac{1}{2}Z_0 e^{C_1\phi}\phi^{;\mu}\phi_{;\mu} - U_0 e^{(C_1-2n)\phi} + L_{\text{fluid}}\right]. \tag{91}$$

As in the above cases, from the conservation equation $\rho = \rho_0 a^{-3(\gamma+1)}$ and the first of the FE Eq. (7), we obtain that $-3a_1(\gamma+1) = (m-2n)/n$, so

$$a_1 = \frac{2n-m}{3n(\gamma+1)}, \tag{92}$$

and therefore

$$q = \frac{3n(\gamma+1)}{2n-m} - 1, \tag{93}$$

thus, by considering that $a_1 > 0$, $q < 0$ and U decreasing, we may find some restrictions on the free parameters n and m ($\gamma \in (-1, 1]$). In general, we found that $m < 0$, $0 < n$ such that $\gamma < -(n+m)/3n$. If we set $m = 0$, then, $\gamma \in (-1, -1/3)$ while n is free. If $m = -1$, then $n < -1/2$ or $0 < n$ with $\gamma < -(n+1)/3n$.

5. Lagrangian Collineation Approach

The Lagrangian density takes the following form (without the matter field, that is $\rho = 0 = p$):

$$\mathcal{L} = a^3(FR - f) + 6a\dot{a}^2F + 6a^2\dot{a}\dot{F} - a^3(Z\dot{\phi}^2 - 2U), \tag{94}$$

or

$$\mathcal{L} = ga^3(Rh_R - h) + 6ga\dot{a}^2h_R + 6ga^2\dot{a}\dot{R}h_{RR} + 6g_\phi a^2\dot{a}\dot{\phi}h_R - a^3(Z\dot{\phi}^2 - 2U), \tag{95}$$

where $F = f_R = \partial_R f(R,\phi)$, and $\dot{F} = \dot{R}\partial_{RR}f + \dot{\phi}\partial_{R\phi}f$.

We consider in this approach the Lagrangian (95) so $Q = (a, \phi, R)$ and therefore $TQ = (a, \dot{a}, \phi, \dot{\phi}, R, \dot{R})$. As we can see, the Hessian is different of zero iff

$$\mathcal{H} = 72g^2h_{RR}^2a^7Z \neq 0 \quad\Leftrightarrow\quad h_{RR} \neq 0. \tag{96}$$

FE reads

$$3FH^2 = \frac{Z}{2}\dot{\phi}^2 - 3H\dot{F} + \frac{1}{2}(RF - f) + U, \tag{97}$$

$$F(2\dot{H} + 3H^2) = -\frac{Z}{2}\dot{\phi}^2 - 2H\dot{F} - \ddot{F} + \frac{1}{2}(RF - f) + U, \tag{98}$$

$$2Z(\ddot{\phi} + 3H\dot{\phi}) = f_\phi - Z_\phi\dot{\phi}^2 - 2U_\phi. \tag{99}$$

Let X be a VF, such that

$$\mathbf{X} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial\phi} + \gamma\frac{\partial}{\partial R} + \dot{\alpha}\frac{\partial}{\partial\dot{a}} + \dot{\beta}\frac{\partial}{\partial\dot{\phi}} + \dot{\gamma}\frac{\partial}{\partial\dot{R}}, \tag{100}$$

where α, β and γ are functions of the scale factor a, the scalar field ϕ and R. We say that the vector field, $\mathbf{X}$, is a Lagrangian collineation if it satisfies the equation

$L_{\mathbf{X}}\mathcal{L} = \mathbf{X}\mathcal{L} = 0$, where $L_{\mathbf{X}}$ stands for the Lie derivative with respect to the vector field $\mathbf{X}$. We find the associated system of PDE

$$\begin{aligned}&6\alpha g h_R + 6a\beta g_\phi h_R + 6a\gamma g h_{RR} + 12a\alpha_a g h_R \\ &\qquad + 6\beta_a g_\phi h_R a^2 + 6g h_{RR}\gamma_a a^2 = 0, \end{aligned} \tag{101}$$

$$6\alpha_\phi g_\phi h_R - 2Z\beta_\phi a - \beta Z_\phi a - 3Z\alpha = 0, \tag{102}$$

$$6g h_{RR} a^2 \alpha_R = 0, \tag{103}$$

$$\begin{aligned}&Z\beta_a a^2 - 6\alpha g_\phi h_R - 3\beta g_{\phi\phi} h_R a - 3\gamma g_\phi h_{RR} a - 3g_\phi h_R a\alpha_a \\ &\qquad - 6g h_R \alpha_\phi - 3g_\phi h_R a\beta_\phi - 3g h_{RR}\gamma_\phi a = 0,\end{aligned} \tag{104}$$

$$\begin{aligned}&2\alpha g h_{RR} + \beta g_\phi h_{RR} a + \gamma g h_{RRR} a + g h_{RR} a\alpha_a \\ &\qquad + 2g h_R \alpha_R + g_\phi h_R a\beta_R + g h_{RR}\gamma_R a = 0,\end{aligned} \tag{105}$$

$$3\alpha_R g_\phi h_R - \beta_R Z a + 3g h_{RR}\alpha_\phi = 0, \tag{106}$$

$$\begin{aligned}&-3g h_R \alpha R + 3g h\alpha - g_\phi h_R \beta R a + h g_\phi \beta a \\ &\qquad - \gamma g a R h_{RR} - 6\alpha U - 2\beta a U_\phi = 0.\end{aligned} \tag{107}$$

We would like to point out that in [35], the authors have obtained a bit different system of PDE.

We have found the following solution

$$\begin{aligned}&\alpha = c_1 a, \quad \beta = \varphi(\phi) \quad \gamma = 0, \quad g = m e^{\int \frac{-3c_1}{\varphi(\phi)} d\phi}, \\ &h = R^{\tilde{n}}, \quad Z = c_2 e^{\int \frac{-3c_1 + 2\varphi'(\phi)}{\varphi(\phi)} d\phi}, \quad U = c_3 e^{\int \frac{-3c_1}{\varphi(\phi)} d\phi},\end{aligned} \tag{108}$$

where, $(c_i)_{i=1}^3, m$, are constants of integration and $\varphi'(\phi) = \frac{d\varphi}{d\phi}$, being $\varphi(\phi)$ an arbitrary function of ϕ. This means that for different choices of $\varphi(\phi)$, we will obtain different solutions. For example, by assuming $\varphi(\phi) = -\frac{3c_1}{n}\phi$, we obtain

$$\begin{aligned}&\alpha = c_1 a, \quad \beta = -\frac{3c_1}{n}\phi, \quad \gamma = 0, \quad g = m\phi^n, \\ &h = R^{\tilde{n}}, \quad Z = Z_0\phi^{n-2}, \quad U = U_0\phi^n,\end{aligned} \tag{109}$$

and therefore the action collapses to

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R^r\phi^n - \frac{1}{2}Z_0\phi^{n-2}\phi^{;\mu}\phi_{;\mu} - U_0\phi^n + \mathcal{L}_m\right]. \tag{110}$$

But, if we assume $\varphi(\phi) = -3c_1/n$, then we obtain

$$\begin{aligned}&\alpha = c_1 a, \quad \beta = -\frac{3c_1}{n}, \quad \gamma = 0, \quad g = g_0 e^{n\phi}, \\ &h = R^{\tilde{n}}, \quad Z = Z_0 e^{n\phi}, \quad U = U_0 e^{n\phi},\end{aligned} \tag{111}$$

and therefore the action collapses to

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R^r e^{n\phi} - \frac{1}{2}Z_0 e^{n\phi}\phi^{;\mu}\phi_{;\mu} - U_0 e^{n\phi} + \mathcal{L}_m\right], \tag{112}$$

which is similar but not identical to the obtained one in Eq. (91), that is, the solution generated by the symmetry $[at, b]$.

Remark 2. From the results (109), we note that the invariant solution induced by the symmetry $\mathbf{X}$, that is, $\alpha = c_1 a$, $\beta = -3c_1\phi/n$, and $\gamma = 0$, is the following one:

$$-\frac{3da}{na} = \frac{d\phi}{\phi} = \frac{dR}{0},$$

thus

$$\phi = a^{-3/n}, \quad \text{or } a = a_0\phi^{-n/3}, \quad \text{and} \quad R = \text{const},$$

arriving at the same conclusions through dimensional considerations, since from the FE

$$3FH^2 = \frac{Z}{2}\dot{\phi}^2 - 3H\dot{F} + \frac{1}{2}(RF - f) + U,$$

with

$$g = m\phi^n, \quad h = R^{\tilde{n}}, \quad Z = Z_0\phi^{n-2}, \quad U = U_0\phi^n,$$

we may observe that

$$[U] = [g] = [f] \;\Rightarrow\; [h] = 0,$$

and therefore

$$R = \text{const}, \Rightarrow \quad H = \text{const} \quad \Rightarrow \quad a = \exp(a_1 t).$$

By taking into account the results (109), the Lagrangian (95) yields

$$\mathcal{L} = m\tilde{n}R^{\tilde{n}-1}\phi^n a\left(\frac{(\tilde{n}-1)}{\tilde{n}}Ra^2 + 6\dot{a}^2 + 6(\tilde{n}-1)a\dot{a}\dot{R}R^{-1} + 6n\phi^{-1}a\dot{a}\dot{\phi}\right) - a^3\phi^n(Z_0\phi^{-2}\dot{\phi}^2 - 2U_0), \tag{113}$$

thus, the conserved quantity, $\Sigma = i_X\theta_{\mathcal{L}}$, where $\theta_{\mathcal{L}} = \partial_{\dot{a}}\mathcal{L}da + \partial_{\dot{\phi}}\mathcal{L}d\phi + \partial_{\dot{R}}\mathcal{L}dR$, yields

$$\Sigma = i_X\theta_{\mathcal{L}} = \alpha\partial_{\dot{a}}\mathcal{L} + \beta\partial_{\dot{\phi}}\mathcal{L}, \tag{114}$$

since $\alpha = c_1 a$, $\beta = -3c_1\phi/n$, and $\gamma = 0$, then Σ collapses to

$$\Sigma = 6m\tilde{n}\phi^n a^2 R^{\tilde{n}-1}(-\dot{a} + (\tilde{n}-1)a\dot{R}R^{-1} + n\phi^{-1}a\dot{\phi}) + \frac{6}{n}Z_0 a^3\phi^{n-1}\dot{\phi}, \tag{115}$$

now, by assuming, $\Sigma = 0$:

$$m\tilde{n}R^{\tilde{n}-1}(-\dot{a} + (\tilde{n}-1)a\dot{R}R^{-1} + n\phi^{-1}a\dot{\phi}) = -\frac{1}{n}Z_0 a\phi^{-1}\dot{\phi}, \tag{116}$$

note that $R = 6(\dot{H}^2 + 2H^2) = \text{const}$, and if we take into account our previous result, $\dot{R} = 0$, then

$$-nm\tilde{n}R^{\tilde{n}-1}\frac{\dot{a}}{a} = -n^2 m\tilde{n}R^{\tilde{n}-1}\phi^{-1}\dot{\phi} - Z_0\phi^{-1}\dot{\phi}, \tag{117}$$

in this way

$$A\frac{\dot{a}}{a}=\left(An+\frac{Z_0}{n}\right)\frac{\dot{\phi}}{\phi},\quad A=m\tilde{n}R_0^{\tilde{n}-1},\tag{118}$$

and therefore

$$a=a_0\phi^l,\quad l=n+\frac{Z_0}{An},\tag{119}$$

which is the result obtained previously by considering the invariant solution.

5.1. *Friedman equation*

We now study the first of the FE in order to determine the scalar function, ϕ, therefore by considering

$$3FH^2=\frac{Z}{2}\dot{\phi}^2-3H\dot{F}+\frac{1}{2}(RF-f)+U,\tag{120}$$

with $g=m\phi^n$, $h=R^{\tilde{n}}$ (such that $\dot{R}=0$), $Z=Z_0\phi^{n-2}$, $U=U_0\phi^n$, with $h_{RR}\neq 0$, then (algebra brings us to obtain)

$$3\tilde{n}mH^2=\frac{Z_0}{2R^{\tilde{n}-1}}\frac{\dot{\phi}^2}{\phi^2}-3\tilde{n}mnH\frac{\dot{\phi}}{\phi}+\frac{m}{2}R(\tilde{n}-1)+\frac{U_0}{R^{\tilde{n}-1}}.\tag{121}$$

By taking into account the relationship $a=a_0\phi^{-n/3}$, then $H=-n\dot{\phi}/3\phi$, and therefore

$$\frac{\tilde{n}mn^2}{3}\frac{\dot{\phi}^2}{\phi^2}=\frac{Z_0}{2R^{\tilde{n}-1}}\frac{\dot{\phi}^2}{\phi^2}-3\tilde{n}mnH\frac{\dot{\phi}}{\phi}+\frac{m}{2}R(\tilde{n}-1)+\frac{U_0}{R^{\tilde{n}-1}},\tag{122}$$

and we also consider that $R=\text{const.}$ so

$$A\dot{\phi}^2+B\phi^2=0,\tag{123}$$

with

$$A=\left(\frac{2}{3}\tilde{n}mn^2+\frac{Z_0}{2R^{\tilde{n}-1}}\right),\quad B=\frac{m}{2}R(\tilde{n}-1)+\frac{U_0}{R^{\tilde{n}-1}}=\text{const.}\tag{124}$$

Equation (123) admits the following solution

$$\phi=\phi_0 e^{\pm\sqrt{-\frac{B}{A}}t},\tag{125}$$

which is in agreement with the second solution obtained through the Lie group method.

6. Conclusions

We have studied the $f(R,\phi)$ cosmological models with a flat FRW metric through three different symmetry methods and by making an assumption on the form of the function $f(R,\phi)=h(R)g(\phi)$. In this way, we have shown how to generate exact analytical solutions and how they could explain the observed acceleration of our universe.

With the first of the approaches, matter collineation, we have shown that this kind of models admits SS solutions by determining the form of each of the unknown functions, finding two different solutions. The method allows us to determine the exact expressions for the functions Z and U in function of t as well as in function of ϕ. Although we have worked with a FRW metric, the obtained results with this technique can be generalized to anisotropic metric (the SS Bianchi models) as well as for the Kantowski–Sach metric, since, in these cases, the scale factors follow a power law solution.

With the method of the Lie groups, the most general of all of them, we have obtained three exact solutions, as examples, but more solutions could be obtained following the receipt, that is, by imposing other symmetries. The first of the solutions is identical to the first solution obtained through the matter collineation approach and therefore it is also SS. We have determined that the function f must be of the form $f(R,\phi)=R^r\phi^n$, in such a way that the obtained model generalizes the well-known model of the Brans–Dicke, induced gravity, etc, for different values of n, but with R^r. In the same way, we have shown how to constrain the parameters, (n,m,γ) in order to obtain accelerating solutions and a decreasing potential on time (it mimics a variable cosmological constant). In the second example, we have obtained a de Sitter like solution and therefore R, the Ricci scalar, is constant. In this case, we have obtained a new cosmological solution, in the best of our knowledge, by studying the wave function for the scalar field. We also have studied the invariant, particular, solution induced by the symmetry. We have emphasized that this solution is not valid for $m=-2$. The third solution corresponds to the second SS solution. Nevertheless, in our opinion, the followed method have a drawback. Note that we have studied only the wave equation (WE) ignoring the rest of the FE, and therefore by imposing a symmetry, we are able to determine the form of the function f, Z and U in order to obtain a solution, but only for this equation (the WE), thus one needs to check that this solution is also a solution of the rest of the FE. In the exposed examples, we have checked that the solution is a solution of the FE not only of the WE, so not all symmetry brings us to obtain a complete solution of the FE. In fact, the FE reduce to a relation among constants, that is, they reduce to an algebraic system of equations. This system of equations has three equations and two auxiliary relationships; $a_1=a_1(n,m,\gamma)$, and $R_0=R_0(a_1)$, with eight unknowns $(\phi_0,\rho_0,g_0,Z_0,U_0,m,n,r)$, thus, we can only determine three constants in function of the other five. These five undefined constants can be constrained by physical arguments (the potential U must be decreasing, the energy density must be positive and decreasing) or by taking into account observational data (the cosmological constant must be positive, that is, $U_0>0$, or the deceleration parameter must be negative, $q<0$).

With regard to the Lagrangian collineation approach, we have only been able to obtain a solution to the equation $L_{\mathbf{X}}\mathcal{L} = 0$, but this kind of equations admits more solutions. Nevertheless, our solution depends on a function $\varphi = \varphi(\phi)$ which allows us to generate many solutions. We have shown two cases. The first case given by Eq. (109) coincides with the second solution generated with the Lie group method. The second case (111) is similar but not identical to the obtained one in Eq. (91), that is, the solution generated by the symmetry $[at, b]$.

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