Exact solutions of a full causal bulk viscous FRW cosmological model with variable G and Λ through factorization

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We study the classical flat full causal bulk viscous FRW cosmological model with variable gravitational and cosmological constants through the factorization method. The method allows us to find some new exact parametric solutions. The assumptions made bring us to study two approaches. We find, in the studied cases, that the Universe ends in an accelerating era except in the case of a particular solution where the Universe could be noninflationary for all times. In both approaches the cosmological constant is a decreasing function of time, while the gravitational constant behaves as a growing or decreasing time function depending on the sign of Λ. By taking into account recent observations that indicate that Λ must be positive, we conclude that G increases with time except in the first solution where both "constants" tend asymptotically in the large time limit to a constant value. We also present a new factorization scheme which allows us to generate new solutions to a kind of variable coefficient nonlinear second order ODE.

Keywords: Full causal bulk viscosity, FRW cosmologies, time-varying constants, Factorization method, exact solutions.

I. INTRODUCTION

The distribution of matter can be satisfactorily described by a perfect fluid due to the large scale distribution of galaxies in our universe. However, observed physical phenomena such as the large entropy per baryon and the remarkable degree of isotropy of the cosmic microwave background radiation, suggest analysis of dissipative effects in cosmology. Furthermore, there are several processes which are expected to give rise to viscous effects. These are the decoupling of neutrinos during the radiation era and the decoupling of radiation and matter during the recombination era. Bulk viscosity is associated with the GUT phase transition and string creation. Misner [1–3] and Weinberg [4, 5] have studied the effect of viscosity on the evolution of cosmological models. Due to such assumption, dissipative processes are supposed to play a fundamental role in the evolution of the early Universe. The theory of relativistic dissipative fluids, created by Eckart [7] and Landau and Lifshitz [8] has many drawbacks, and it is known that it is incorrect in several aspects mainly those concerning causality and stability. Israel [9] formulates a new theory in order to solve these drawbacks. This theory was later developed by Israel and Stewart [10] into what is called transient or extended irreversible thermodynamics. The best currently available theory for analyzing dissipative processes in the Universe is the full causal thermodynamics developed by Israel and Stewart [10], Hiscock and Lindblom [11] and Hiscock and Salmonson [12]. The full causal bulk viscous thermodynamics has been extensively used to study the evolution of the early Universe and some astrophysical processes [13, 14]. However, due to the complicated nature of the evolution equations, very few exact cosmological solutions of the gravitational field equations are known in the framework of the full causal theory [15]. Recently, several authors have studied the possibility that a single imperfect fluid with bulk viscosity can replace the need for separate dark matter and dark energy in cosmological models. Since bulk viscosity implies negative pressure, this rises the possibility of unifying the dark sector. With suitable choices of model parameters, it has been shown that the background cosmology in these models can mimic that of a ΛCDM Universe to high precision, since the presence of dissipative effects could alleviate some of the problems presented in the ΛCDM model. In the same way, these kinds of models, in the homogeneous and isotropic background have similarities with a generalized Chaplygin gas model (see for instance [16–22]).

Since the pioneering proposal of Dirac [23] on a model with a time variable gravitational coupling constant G, motivated by the occurrence of large numbers in the universe or numerological coincidences uncovered by Weyl, Eddington and Dirac himself (see for instance [24] for a review and references therein), cosmological models with variable G have been intensively investigated in the literature, as for example Jordan-Brans-Dicke model and its generalizations [25] or more recently the models proposed by Lu et al [26] or Smolin [27]. In the same way, there are in the literature many works devoted to studying the possible variations of G through astrophysical and cosmological observations (see for instance [28] for a review and references therein). On the other hand, recent developments

in particle physics and cosmology have shown that the cosmological constant Λ ought to be treated as a dynamical quantity [29] rather than a simple constant, in order to alleviate the so called fine-tuning and coincidence problems [30]. The dynamics of the scale factor in FRW type models with a variable Λ term has recently been revisited by Overduin and Cooperstock [31] for the perfect fluid case. Alternative models of dark energy suggest a dynamical form of dark energy originating from a variable cosmological constant $\Lambda(t)$ (see Sola et al [32]). Since the cosmological models based on General Relativity do not allow any possible variation in the gravitational constant G and cosmological constant Λ, because of the fact that the Einstein and stress-energy tensors have zero divergence, then it is necessary to introduce some modifications to the Einstein equations, in order to avoid violations of the energy conservation law.

The effects of dissipation, as expressed in the form of a non-vanishing bulk viscosity coefficient in the stress-energy tensor of matter in cosmological models with variable Λ and G, have been considered by several authors (we cite only a few of them [33–40]). In [39] the authors present an unified description of the early evolution of the universe with a number of possible assumptions on the bulk viscous term and the gravitational constant, in which an inflationary phase is followed by radiation-dominated phase. They also show that the effect of viscosity affects the past and future of the universe. Arbab [40] has shown that the cosmological and gravitational constants increase exponentially with time, whereas the energy density and viscosity decrease exponentially with time. The rate of mass creation during inflation is found to be huge suggesting that all matter in the universe is created during inflation. But only a few works can be found within the framework of the full causal theory (see for instance [41, 42]).

Therefore, the purpose of the present work is to study the full causal bulk viscous cosmological model with flat FRW symmetries, allowing that G and Λ may vary with the time. In order to find some new exact solutions, we employ the factorization method since it has been very useful for this purpose in a previous work in this context [15].

The factorization of linear second order differential equations is a well-established method in finding exact solutions through algebraic procedures. It was first introduced by Dirac to solve the spectral problem for the quantum oscillator, and some years later, had a further development due to Schrodinger's works on the factorization of the Sturm-Liouville equation. At the present time, very good informative reviews on the factorization method can be found in the open literature (see for instance [43, 44]). Recently, the factorization method has been applied to find exact solutions of nonlinear ordinary differential equations (ODE) [45–52]. In [46] and [47], a systematic way to apply the factorization method to nonlinear second order ODE has been provided. The factorization of some ODE may be restricted due to constraints which appear in a natural way within the factorization procedure. In this work, we have been able to get exact parametric solutions of some ODE which do not allow their factorization or present cumbersome constraints, by simply performing transformation of coordinates. Also, a new factorization scheme has been developed to solve a kind of variable coefficient nonlinear ODE for the energy density, which raises in the proposed model.

The paper is organized as follows. In Section II, we start by reviewing the main components of a flat bulk viscous FRW cosmological model with variable G and Λ, and introduce the assumptions. These assumptions bring us to study two different approaches. In Section III we study different cases of the first approach while Section IV is devoted to the study of the second one. Finally, we summarize our conclusions in Section V.

II.   THE MODEL.

The Einstein gravitational field equations (FE) with variable G and Λ are:

$$R_{ij} - \frac{1}{2}g_{ij}R = 8\pi G(t)T_{ij} + \Lambda(t)g_{ij}, \tag{1}$$

in the following we consider a system of units so that $c = 1$. The bulk viscous effects can be generally described by means of an effective pressure Π, formally included in the effective thermodynamic pressure p_{eff} [14]. Then, in the comoving frame the energy momentum tensor has the components $T_0^0 = \rho, T_1^1 = T_2^2 = T_3^3 = -p_{eff}$. In the presence of the bulk viscous stress Π, the effective thermodynamic pressure term becomes $p_{eff} = p + \Pi$, where p is the thermodynamic pressure of the cosmological fluid. The causal evolution equation for the bulk viscous pressure is given by [14]

$$\tau\dot{\Pi} + \Pi = -3\xi H - \frac{1}{2}\tau\Pi\left(3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\right), \tag{2}$$

where T is the temperature, ξ the bulk viscosity coefficient and τ the relaxation time. We have first to give the equation of state for p and specify T, τ and ξ. We shall assume the following laws [14]:

$$p = (\gamma - 1)\rho, \quad \xi = \alpha\rho^s, \quad T = T_0\rho^{\frac{\gamma-1}{\gamma}}, \quad \tau = \frac{\xi}{\rho} = \alpha\rho^{s-1}, \tag{3}$$

where $1 \leq \gamma \leq 2$, $\alpha \geq 0$, $T_0 \geq 0$ and $s \geq 0$ are constants. Equations (3) are standard in the study of bulk viscous cosmological models, whereas the equation $\tau = \xi/\rho$ is a simple procedure to ensure that the speed of viscous pulses does not exceed the speed of light.

In order to take into account the variations of G and Λ we use the Bianchi identities (see for instance [53–55])

$$\left(R_{ij} - \frac{1}{2} R g_{ij} \right)^{;j} = 0 = \left(8\pi G T_{ij} - \Lambda g_{ij} \right)^{;j}, \tag{4}$$

which read:

$$8\pi G \left[\dot{\rho} + (\rho + p_{eff}) \theta \right] = -\dot{\Lambda} - 8\pi \dot{G} \rho, \tag{5}$$

(a dot means time derivative) and $\theta = u^i_{;i}$ is the expansion factor, where u^i is the 4-velocity. Assuming that the total matter content of the Universe is conserved, $T^j_{i;j} = 0$, then the energy conservation equation can be split into two independent equations:

$$\dot{\rho} + (\rho + p_{eff}) \theta = 0, \quad \text{and} \quad \dot{\Lambda} = -8\pi \dot{G} \rho. \tag{6}$$

As it is observed, from the equation: $\dot{\Lambda} = -8\pi \dot{G} \rho$, the behaviour of Λ and G are related in such a way that when G is growing then Λ is positive, but if G is decreasing then Λ is negative. If G behaves as a true constant, then Λ vanishes.

We consider that the geometry of the Universe can be described by a spatially flat FRW type metric given by

$$ds^2 = dt^2 - f^2(t) \left[dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right]. \tag{7}$$

Therefore, for the line element (7), the gravitational field equations describing the cosmological evolution of a causal bulk viscous fluid in presence of the variable gravitational and cosmological constants are

$$3H^2 = 8\pi G(t)\rho + \Lambda(t), \tag{8}$$

$$2\dot{H} + 3H^2 = -8\pi G(t) \left(p + \Pi \right) + \Lambda(t), \tag{9}$$

$$\dot{\rho} + 3 \left(\rho + p + \Pi \right) H = 0, \tag{10}$$

$$\dot{\Lambda}(t) = -8\pi \rho \dot{G}(t), \tag{11}$$

$$\tau \dot{\Pi} + \Pi = -3\xi H - \frac{\epsilon}{2} \tau \Pi \left(3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T} \right). \tag{12}$$

An important observational quantity is the deceleration parameter $q = \frac{d}{dt} \left(\frac{1}{H} \right) - 1$, where $H = \dot{f}/f$. The sign of the deceleration parameter indicates whether the model inflates or not. The positive sign of q corresponds to "standard" decelerating models whereas the negative sign indicates inflation.

The growth of the total comoving entropy $\Sigma(t)$ over a proper time interval (t_0, t) is

$$\Sigma(t) - \Sigma(t_0) = -\frac{3}{k_B} \int_{t_0}^{t} \Pi f^3 H T^{-1} dt, \tag{13}$$

where k_B is the Boltzmann constant.

A. Assumptions

Approach 1. With the previous choices, the general solution of the gravitational field equations with variable gravitational and cosmological constants still depends on the functional form of G and Λ [41]. In the first approach, we shall fix the mathematical form of the cosmological constant assuming that it is a function of the Hubble parameter only, and its time dependence is [34]:

$$\Lambda = 3\beta H^2, \tag{14}$$

the gravitational constant G and the energy density ρ are given by:

$$G = bH^{-n\beta}, \quad \rho = \rho_0 H^n, \tag{15}$$

where $\rho_0 = 3/4\pi bn \geq 0$ and $b \geq 0$ are constants, and $n = 2/(1-\beta) \geq 0$, in order to assure a time decreasing energy density of the Universe. One must also assume that H is a decreasing function of the cosmological time.

With the use of the barotropic equation of state $p = (\gamma - 1)\rho$ and Eq. (9), we obtain

$$\dot{H} + \frac{3\gamma}{n}H^2 + 4\pi G\Pi = 0. \tag{16}$$

In view of Eq. (15), Eq. (16) becomes

$$\dot{H} + \frac{3\gamma}{n}H^2 + 4\pi bH^{-n\beta}\Pi = 0. \tag{17}$$

With the use of Eqs. (3) and (17), the causal evolution equation for the bulk viscosity (12) leads to the following equation for the Hubble function H:

$$\ddot{H} + \left[3H + \alpha_0 H^{n(1-s)}\right]\dot{H} + \left(\frac{n - 4\gamma}{2\gamma}\right)H^{-1}\dot{H}^2 + \frac{9}{n}\left(\frac{\gamma}{2} - 1\right)H^3 + \frac{3\gamma\alpha_0}{n}H^{2+n(1-s)} = 0, \tag{18}$$

where $\alpha_0 = \rho_0^{1-s}/\alpha$.

Approach 2 [42]. From the field equations we obtain for the derivative of the Hubble function the alternative expression

$$\dot{H} = -4\pi G(t)\left(\rho + p + \Pi\right). \tag{19}$$

As it has been pointed out by R. Maartens in [14], all the relationships between the physical quantities are phenomenological or imposed ad hoc so that the principles of thermodynamics are not infringed. If we rewrite Eq. (2) by taking into account Eq. (3), then we obtain an equation that expresses the bulk viscous pressure as a function of the energy density only. Now, as it has been pointed out in Ref. [42], the dimensional analysis shows us that

$$[\Pi] = [p] = [\rho], \tag{20}$$

that is, these physical quantities have the same dimensional equation, so the natural way (from the dimensional point of view) is to set an equation of state of the form

$$\Pi = -\chi\rho, \tag{21}$$

where $\chi \geq 0$.

With the last assumption, the bulk viscosity evolution equation can be rewritten in the alternative form

$$\frac{1}{2\gamma}\frac{\dot{\rho}}{\rho} + \frac{1}{\alpha}\rho^{1-s} = 3\left(\frac{1}{\chi} - \frac{1}{2}\right)H. \tag{22}$$

Taking the derivative with respect to the time of this equation and with the use of Eq. (19), we obtain the following second order differential equation describing the time variation of the density of the cosmological fluid:

$$\ddot{\rho} - \frac{1}{\rho}\dot{\rho}^2 + D\rho^{1-s}\dot{\rho} - AG(t)\rho^2 = 0, \tag{23}$$

where $D = 2(1-s)\gamma/\alpha > 0$ and $A = 12\pi(\gamma - \chi)(\chi - 2)\gamma/\chi < 0$.

III. FACTORIZATION METHOD FOR APPROACH 1

A. Case 1

The Eq. (18) can be simplified by performing the following transformation of the dependent and independent variables,

$$H = y^a, \qquad d\eta = Ky^a dt, \tag{24}$$

where a and K are constants. Then, it turns into the ODE

$$y'' + \left(-1 + \frac{an}{2\gamma}\right)\frac{1}{y}y'^2 + \frac{(3 + \alpha_0 y^m)}{K}y' + \frac{9}{anK^2}\left(\frac{\gamma}{2} - 1\right)y + \frac{3\gamma\alpha_0}{anK^2}y^{m+1} = 0, \tag{25}$$

where $y' = \frac{dy}{d\eta}$, and $m = a\left(n(1-s) - 1\right)$. This ODE can also be rewritten as follows

$$y'' + A_1 \frac{1}{y} y'^2 + \frac{(3 + \alpha_0 y^m)}{K} y' + (C_1 + D_1 y^m) y = 0, \tag{26}$$

where

$$A_1 = -1 + \frac{an}{2\gamma}, \quad C_1 = \frac{9}{anK^2}\left(\frac{\gamma}{2} - 1\right), \quad D_1 = \frac{3\gamma\alpha_0}{anK^2}. \tag{27}$$

Let us consider the following factorization scheme [46–48] applied to Eq. (25). The nonlinear second order equation

$$y'' + f(y) y'^2 + g(y)y' + h(y) = 0, \tag{28}$$

where $y' = \frac{dy}{d\eta} = D_\eta y$, can be factorized in the form

$$[D_\eta - \phi_1(y)y' - \phi_2(y)] [D_\eta - \phi_3(y)] y = 0, \tag{29}$$

under the conditions

$$f(y) = -\phi_1, \tag{30}$$

$$g(y) = \phi_1\phi_3 y - \phi_2 - \phi_3 - \frac{d\phi_3}{dy}y, \tag{31}$$

$$h(y) = \phi_2\phi_3 y, \tag{32}$$

which are obtained by developing the differential operators on the dependent variable y in Eq. (29), and by comparing to Eq. (28).

If we assume $[D_\eta - \phi_3(y)] y = \Omega(y)$, then the factorized Eq. (29) can be rewritten as

$$y' - \phi_3 y = \Omega, \tag{33}$$

$$\Omega' - (\phi_1 y' + \phi_2)\Omega = 0. \tag{34}$$

We can introduce the functions ϕ_i by comparing Eqs. (26) (or (25)) and (28). Then, the functions

$$\phi_1 = -\frac{A_1}{y}, \quad \phi_2 = a_1^{-1}, \quad \text{and} \quad \phi_3 = a_1(C_1 + D_1 y^m), \tag{35}$$

where $a_1(\neq 0)$ is an arbitrary constant, are proposed.

The explicit value of a_1 is obtained from Eq. (31). Since

$$g(y) = \phi_1\phi_3 y - \phi_2 - \phi_3 - \frac{d\phi_3}{dy}y = \frac{(3 + \alpha_0 y^m)}{K}, \tag{36}$$

then, we get

$$-A_1 a_1(C_1 + D_1 y^m) - a_1^{-1} - a_1(C_1 + D_1 y^m) - a_1 m D_1 y^m = \frac{(3 + \alpha_0 y^m)}{K}, \tag{37}$$

and by comparing both sides of Eq. (37), the following equations are obtained

$$-A_1 a_1 C_1 - a_1^{-1} - a_1 C_1 = \frac{3}{K}, \tag{38}$$

$$-A_1 a_1 D_1 - a_1 D_1 - a_1 m D_1 = \frac{\alpha_0}{K}, \tag{39}$$

which imply

$$\frac{9}{2\gamma}\left(\frac{\gamma}{2} - 1\right) a_1^2 + 3K a_1 + K^2 = 0, \tag{40}$$

and

$$\frac{3\gamma a_1}{anK}\left(\frac{an}{2\gamma} + m\right) + 1 = 0. \tag{41}$$

Therefore, from Eq. (40) we get

$$a_{1\pm} = \frac{2K(-\gamma \pm \sqrt{2\gamma})}{3\gamma - 6}, \qquad \gamma \neq 2, \tag{42}$$

which is singular for $\gamma = 2$. Also, from Eq. (41) we get

$$m = -\frac{an}{6\gamma a_{1\pm}}(3a_{1\pm} + 2K), \tag{43}$$

or

$$m_\pm = \pm\frac{an\sqrt{2}}{2\gamma^{\frac{3}{2}}}, \tag{44}$$

and taking into account that $m = a(n(1-s) - 1)$, then

$$n_\pm = \frac{2\gamma^{\frac{3}{2}}}{2\gamma^{\frac{3}{2}}(1-s) \mp \sqrt{2}}. \tag{45}$$

It is noted that n_- is always positive, while n_+ could be singular for some values of γ and s.

For the chosen functions ϕ_i, the Eq. (34) provides

$$\Omega = \kappa_1 e^{\eta/a_1} y^{-A_1}, \tag{46}$$

and therefore, from Eq. (33) we obtain the nonlinear first order differential equation

$$y' - a_1(C_1 + D_1 y^m)y = \kappa_1 e^{\eta/a_1} y^{-A_1}, \tag{47}$$

where κ_1 is an integration constant. The authors have not been able to find the general solution of Eq. (47). However, particular solutions of Eq. (47) can be found by setting $\kappa_1 = 0$. Then, we get

$$y(\eta) = \frac{1}{\left(\kappa_2 e^{-a_{1\pm} C_1 m\eta} - \kappa_3\right)^{1/m}}, \tag{48}$$

where κ_2 is an integration constant, and

$$\kappa_3 = \frac{D_1}{C_1} = \frac{2\gamma}{3\gamma - 6} < 0, \tag{49}$$

with C_1 and D_1 given in Eq. (27), and $\alpha_0 = 1$. We choose the constant K in such a way that $-a_{1\pm}C_1 = 1$, which provides

$$K_\pm = \frac{3\left(\gamma \mp \sqrt{2\gamma}\right)}{an} = \frac{3\left(\gamma \mp \sqrt{2\gamma}\right)\left(2\gamma^{\frac{3}{2}}(1-s) \mp \sqrt{2}\right)}{a2\gamma^{\frac{3}{2}}}. \tag{50}$$

The time variable is given in parametric form by the integral

$$t(\eta) = \int K^{-1} y^{-a} d\eta. \tag{51}$$

Therefore, the main dynamical variables are given in parametric form as follows:

$$\begin{aligned}
H &= y^a, \\
q &= Ky^a \frac{d}{d\eta}\left(\frac{1}{H}\right) - 1, \\
\Lambda &= 3\beta H^2, \qquad \beta = 1 - 2/n, \\
G &= bH^{-n\beta}, \qquad b > 0 \\
\rho &= \rho_0 H^n, \\
\Pi &= \frac{-\left(Ky^a H' + \frac{3}{2}H^2\right)}{4\pi G}, \\
T &= T_0 \rho^{\frac{\gamma-1}{\gamma}} = y^{\frac{an(\gamma-1)}{\gamma}}, \\
f &= \exp(\eta), \\
\Sigma &= -3\int \Pi f^3 H T^{-1} (Ky^a)^{-1} d\eta.
\end{aligned} \tag{52}$$

In order to analyze this solution, we have plotted the Eq. (52) for a_{1-}, m_-, n_-, K_- and for different values of s and γ. We have set the following numerical values for the constant parameters: $b = 0.4$, $\rho_0 = 3/(4\pi bn)$, $a = 1$, $k_2 = 0.01$, $\gamma = 1.2$ and $s = 0.5$ (solid line), $\gamma = 4/3$ and $s = 0.8$ (dotted line), $\gamma = 1.6$ and $s = 0.85$ (dashed line), and $\gamma = 1.5$ and $s = 0.9$ (long dashed line). Eq. (52) is a particular solution of the most general equation (without any assumption) represented by Eq. (25). The solution obtained is valid for $s \in [0,1]$ and $\gamma \in [1,2)$ (since $C_1 \neq 0$). The energy density, $\rho(t)$, of the cosmological fluid, represented in Fig. (1), is a decreasing function of time but it tends asymptotically to a constant value. The bulk viscous pressure $\Pi(t)$, shown in Fig. (1), satisfies the condition $\Pi < 0$, only for a time interval greater than an initial value of t^*, hence this model can describe the dynamics of the causal bulk viscous Universe with variable gravitational and cosmological constants only for a finite time interval when $t > t^*$. Furthermore, the solution is not thermodynamically consistent since the relationship $|\Pi|/p >> 1$ in all the studied cases. In the era where the viscous effects dominated, a large amount of comoving entropy is produced. Fig. (1) shows the time variation of the entropy. As we can observe, the entropy grows quickly in all the studied cases and in particular when $\gamma = 1.6$ and $s = 0.85$ (dashed curve). With regard to the deceleration parameter $q(t)$, the numerical analysis performed in Fig. (2) shows us that all the studied cases start in a non-decelerating phase, with the deceleration parameter $q > 0$, but it enters into an inflationary phase, with $q = 1$, as the time flows, so the Universe ends in an accelerating epoch. The time variation of the cosmological and gravitational constants is represented in Fig. (2). The cosmological constant is a decreasing function of time but it is positive in the cases: $\gamma = 1.6$ and $s = 0.85$ (dashed curve), and $\gamma = 1.5$ and $s = 0.9$ (long dashed curve), while it is negative in the cases: $\gamma = 1.2$ and $s = 0.5$ (solid curve), and $\gamma = 4/3$ and $s = 0.8$ (dotted curve). Note that basically Λ depends on β, and this constant is positive if $n > 2$ and negative if $n < 2$. In the same way, the gravitational constant $G(t)$, tends in the large time to a constant value, G_0, being growing in the cases: $\gamma = 1.6$ and $s = 0.85$ (dashed curve), $\gamma = 1.5$ and $s = 0.9$ (long dashed curve), while it is decreasing in the cases: $\gamma = 1.2$ and $s = 0.5$ (solid curve), $\gamma = 4/3$ and $s = 0.8$ (dotted curve). Note that $G = H^{(n-2)}$, so it behaves as a decreasing time function when $n < 2$ and growing when $n > 2$, n is given by Eq. (45). Therefore, G and Λ, start varying but they tend to a constant value. The possibility that the cosmological constant and the gravitational coupling are not real constants is an intriguing possibility, which has been intensively investigated in the physical literature. It is a very plausible hypothesis that these effects were much stronger in the early Universe, when dissipative effects also played an important role in the dynamics of the cosmological fluid. Hence, the solutions obtained in the present paper could give an appropriate description of the early period of our Universe.
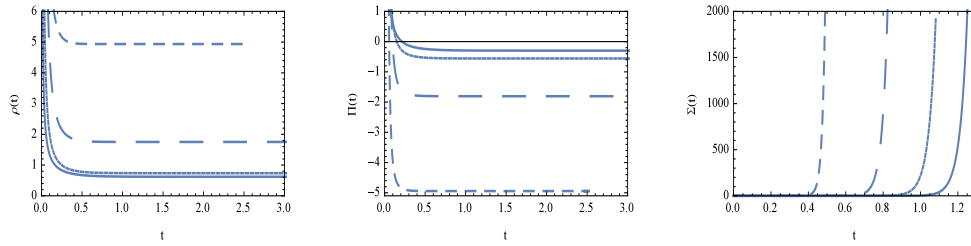

FIG. 1: Plots of the solution (52). Energy density $\rho(t)$, bulk viscous pressure $\Pi(t)$, and the entropy $\Sigma(t)$, for $\gamma = 1.2$ and $s = 0.5$ (solid line), $\gamma = 4/3$ and $s = 0.8$ (dotted line), $\gamma = 1.6$ and $s = 0.85$ (dashed line), $\gamma = 1.5$ and $s = 0.9$ (long dashed line).
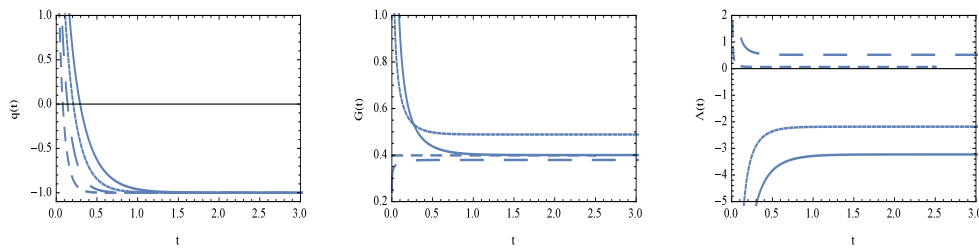

FIG. 2: Plots of the solution (52). $q(t)$, $G(t)$ and $\Lambda(t)$. Parameter values as in Fig. 1.

B. Case 2

We consider now the special case $\gamma = 2$, so Eq. (26) simplifies to the following ODE

$$y'' + A_1 \frac{1}{y} y'^2 + \frac{(3 + \alpha_0 y^m)}{K} y' + D_1 y^{m+1} = 0, \tag{53}$$

where $m = a\left(n(1-s) - 1\right)$, and

$$A_1 = -1 + \frac{an}{4}, \quad D_1 = \frac{6\alpha_0}{anK^2}. \tag{54}$$

Let us assume the factorization scheme provided in Eqs. (28)-(34). The Eq. (53) admits the factorization given in Eq. (29) under the restriction equations

$$f(y) = -\phi_1 = \frac{A_1}{y}, \tag{55}$$

$$g(y) = \phi_1 \phi_3 y - \phi_2 - \phi_3 - \frac{d\phi_3}{dy} y = \frac{(3 + \alpha_0 y^m)}{K}, \tag{56}$$

$$h(y) = \phi_2 \phi_3 y = \frac{6\alpha_0}{anK^2} y^{m+1}. \tag{57}$$

Then, the following factorizing functions ϕ_i

$$\phi_1 = -\frac{A_1}{y}, \quad \phi_2 = a_1^{-1}, \quad \text{and} \quad \phi_3 = a_1 D_1 y^m, \tag{58}$$

where $a_1(\neq 0)$ is an arbitrary constant, are proposed. The explicit value of a_1 is obtained from Eq. (56) which provides

$$\left(-\frac{an}{2\gamma} - m\right) \frac{3\gamma\alpha_0}{anK^2} a_1 y^m - a_1^{-1} = \frac{(3 + \alpha_0 y^m)}{K}. \tag{59}$$

By comparing both sides of Eq. (59), we get

$$a_1 = -\frac{K}{3}, \tag{60}$$

and

$$a_1 = \frac{-2Kan}{3(4m + an)}, \tag{61}$$

then,

$$-\frac{K}{3} = \frac{-2Kan}{3(4m + an)} \quad \Longrightarrow \quad m = \frac{an}{4}, \qquad (\text{if } 2Kan \neq 0). \tag{62}$$

Taking into account the relationship $m = a\left(n(1-s) - 1\right)$, we find the following equation for the parameter n,

$$\frac{an}{4} = a\left(n(1-s) - 1\right) \quad \Longrightarrow \quad n = \frac{4}{3 - 4s}, \tag{63}$$

therefore, $s < 3/4$ in order to get $n > 0$.

The Eq. (34) of the factorization scheme, and the functions ϕ_i (58) generate the explicit form for Ω,

$$\frac{\Omega'}{\Omega} = (\phi_1 y' + \phi_2) = -\frac{A_1}{y} y' + a_1^{-1}, \qquad \Omega = \kappa_1 e^{\eta/a_1} y^{-A_1}, \tag{64}$$

and therefore, from Eq. (33) we obtain the nonlinear first order ODE

$$y' - a_1 D_1 y^m y = \kappa_1 e^{\eta/a_1} y^{-A_1}, \tag{65}$$

where κ_1 is an integration constant.

If $\kappa_1 = 0$, then Eq. (65) yields

$$y' - a_1 D_1 y^{m+1} = 0, \qquad (66)$$

whose solution is given by

$$y(\eta) = (-a_1 D_1 m \eta + \kappa_2)^{-1/m}, \qquad (67)$$

or

$$y(\eta) = \left(\frac{\alpha_0}{2K} \eta + \kappa_2 \right)^{\frac{1}{a}(4s-3)}. \qquad (68)$$

where κ_2 is an integration constant. The parametric form of the time is given by the equation

$$t(\eta) = \int K^{-1} y^{-a} d\eta. \qquad (69)$$

Therefore, we get the main dynamical variables in parametric form as follows

$$
\begin{aligned}
&H = \left(\frac{\alpha_0}{2K} \eta + \kappa_2 \right)^{4s-3}, \\
&f = \exp(\eta), \\
&q = K y^a \frac{d}{d\eta}\left(\frac{1}{H} \right) - 1, \\
&\Lambda = \frac{3}{2}(4s-1)\left(\frac{\alpha_0}{2K} \eta + \kappa_2 \right)^{8s-6}, \qquad \beta = 1 - \frac{2}{n} \in (0,1), \\
&G = b\left(\frac{\alpha_0}{2K} \eta + \kappa_2 \right)^{8s-2}, \qquad b > 0, \\
&\rho = \rho_0 \left(\frac{\alpha_0}{2K} \eta + \kappa_2 \right)^{-4}, \\
&\Pi = \frac{-\left(K y^a H' + \frac{3}{2} H^2 \right)}{4\pi G}, \\
&T = \left(\frac{\alpha_0}{2K} \eta + \kappa_2 \right)^{\frac{2}{3-4s}}, \\
&\Sigma = -3 \int \Pi f^3 H T^{-1} (K y^a)^{-1} d\eta.
\end{aligned}
\qquad (70)
$$

Note that if we set $s = 1/4$, then $\Lambda = 0$ and $G = const$.
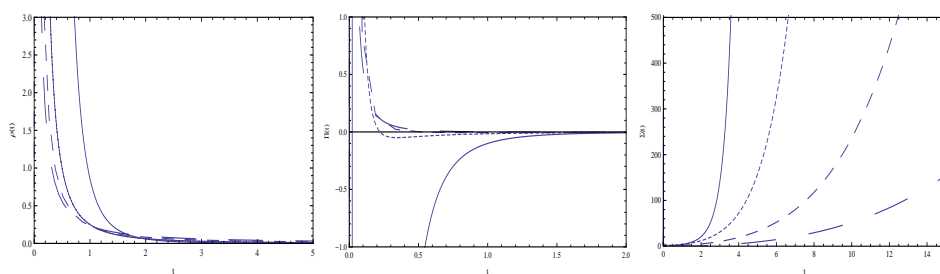

FIG. 3: Plots of the solution (70). Energy density $\rho(t)$, bulk viscous pressure $\Pi(t)$, and the entropy $\Sigma(t)$, for $\gamma = 2$ and different values of s, $s = 0.2$ (long dashed line), $s = 1/3$ (dotted line), $s = 1/2$ (solid line), and $s = 0.74$ (dashed line).

In order to analyze this solution, we have plotted Eq. (70) for $\gamma = 2$ and different values of s, $s = 0.2$ (long dashed line), $s = 1/3$ (dotted line), $s = 1/2$ (solid line), and $s = 0.74$ (dashed line). We have chosen the following values for the constant parameters: $\kappa_2 = 0$, $b = 1$, $\rho_0 = 1$ and $K = 1$.

The solution obtained is valid for $s \in [0, 3/4)$, in order to get $n > 0$, and $\gamma = 2$. The energy density of the cosmological fluid, represented in Fig. (3) is a decreasing function of time. The bulk viscous pressure Π, shown in Fig.
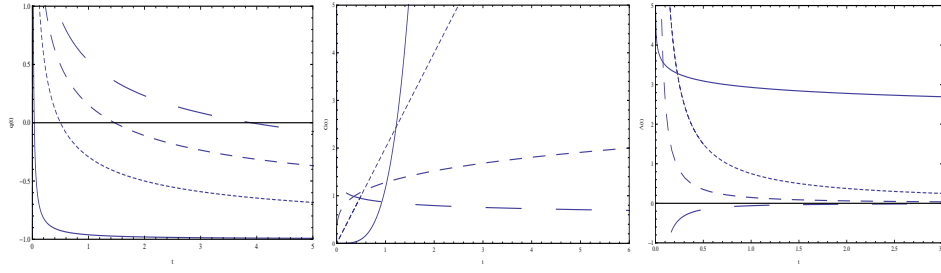
FIG. 4: Plots of the solution (70). $q(t)$, $G(t)$ and $\Lambda(t)$. Parameter values as in Fig. 3.

(3), satisfies the condition $\Pi < 0$, only for time intervals greater than an initial value t_* (cases $s = 0.2$, $s = 1/3$ and $s = 0.74$). Hence, this model can describe the dynamics of the causal bulk viscous Universe with variable gravitational and cosmological constants only for a finite time interval when $t > t_*$. Nevertheless, when $s = 1/2$, the bulk viscous pressure satisfies always the condition $\Pi < 0$, and hence, in this case, the solution can describe the dynamics of the causal bulk viscous Universe with variable gravitational and cosmological constants for $t \in \mathbb{R}^+$. Furthermore, the solution is thermodynamically consistent since the relationship $|\Pi|/p \ll 1$ in all the studied cases.

The behavior of the comoving entropy has been plotted in Fig. (3). As it is observed, when $s = 1/2$, this quantity grows quicker than in the other considered cases.

The evolution of the Universe starts in a non-expansionary phase, with the deceleration parameter $q > 0$ for all the values of s, represented in Fig. (4), but it enters into an inflationary phase as the time flows, so the Universe ends in an inflationary epoch.

The time variation of the cosmological and gravitational constants is represented in Fig. (4). When $s \in (1/4, 3/4)$, the gravitational constant G is always a time growing function while the cosmological constant is a positive decreasing function of time. Only in the case $s = 0.2$ we find that G is decreasing and Λ is negative. As we have pointed out, in the limiting case $s = 1/4$, we get the solution $G = G_0$ and $\Lambda = 0$.

C. Case 3

In this case we fix the value of constant $A_1 = 0$, so $a = \frac{2\gamma}{n}$, and $K = 1$. Then, we get

$$y'' + (3 + \alpha_0 y^m) y' + \frac{9}{2\gamma} \left(\frac{\gamma}{2} - 1 \right) y + \frac{3\alpha_0}{2} y^{m+1} = 0, \tag{71}$$

where $m = \frac{2\gamma[n(1-s)-1]}{n}$. A particular case of this ODE has been studied in [41] for $m = 0$.

Let us consider now the following factorization scheme [49]. The nonlinear second order equation

$$y'' + g(y)y' + h(y) = 0, \tag{72}$$

where $y' = \frac{dy}{d\eta} = D_\eta y$, can be factorized in the form

$$[D_\eta - \phi_2(y)][D_\eta - \phi_1(y)]y = 0, \tag{73}$$

under the conditions

$$\phi_1 + \phi_2 + \frac{d\phi_1}{dy}y = -g(y), \tag{74}$$

$$\phi_1\phi_2 y = h(y). \tag{75}$$

If we assume $[D_\eta - \phi_1(y)]y = \Omega(y)$, and $\phi_2 \equiv const.$, then the factorized Eq. (73) can be rewritten as

$$y' - \phi_1 y = \Omega, \tag{76}$$

$$\Omega' - \phi_2 \Omega = 0. \tag{77}$$

Eq. (77) is easily solved giving as result $\Omega = k_1 e^{\phi_2 \eta}$, where k_1 is an integration constant. Therefore, Eq. (76) is rewritten in the following form

$$y' - \phi_1(y)y = k_1 e^{\phi_2 \eta}, \tag{78}$$

whose solution is also solution of the factorized Eq. (73) with $\phi_2 \equiv const.$

Let us consider the factorizing functions $\phi_1 = a_1(\alpha_1 + \frac{3\alpha_0}{2}y^m)$ and $\phi_2 = a_1^{-1}$. Then, Eq. (71) admits the factorization

$$[D_\eta - a_1^{-1}][D_\eta - a_1(\alpha_1 + \frac{3\alpha_0}{2}y^m)]y = 0, \tag{79}$$

which can be rewritten in the form

$$y' - a_1(\alpha_1 + \frac{3\alpha_0}{2}y^m)y = k_1 e^{a_1^{-1}\eta}. \tag{80}$$

Furthermore, the following relationship is obtained from Eq. (74),

$$a_1\alpha_1 + \frac{3}{2}\alpha_0 a_1 y^m + a_1^{-1} + \frac{3}{2}\alpha_0 a_1 m y^m = -3 - \alpha_0 y^m. \tag{81}$$

Eq. (81) is a noteworthy result which provides the explicit form of a_1 and the relationship among the parameters entering Eq. (71). By comparing both sides of Eq. (81) leads to obtain

$$a_{1\pm}(\gamma) = -\frac{2}{3\left(1 \pm \sqrt{\frac{2}{\gamma}}\right)}. \tag{82}$$

The value $\gamma \neq 2$ avoids singular behavior of a_{1-}. Also, the following relationship for m is obtained,

$$m = -\left(1 + \frac{2}{3a_{1\pm}}\right) = \pm\sqrt{\frac{2}{\gamma}} = \frac{2\gamma[n(1-s)-1]}{n}, \tag{83}$$

and therefore,

$$n_\pm = \frac{\sqrt{2}}{\sqrt{2}(1-s) \mp \gamma^{-3/2}}. \tag{84}$$

One of the advantages of the factorization method as opposed to different approaches is expressed through Eqs. (83) and (84). The last equation provides the relationship between the parameters s, γ and n in such a way that by fixing γ and s we get a particular value of n, thus the method provides the connection among the constant parameters entering the factorized ODE.

1. Solution for $\gamma \neq 2$

We find a particular solution of Eq. (80) by setting $k_1 = 0$, so we consider the ODE

$$y' - (c_0 + c_1 y^m)y = 0, \tag{85}$$

whose solution is given by

$$y(\eta) = \frac{1}{(C_1 e^{-c_1 m\eta} - C_2)^{1/m}}, \tag{86}$$

where C_1 is an integration constant, and

$$c_0 = a_1\alpha_1 = -\frac{3}{2}\left(1 \mp \sqrt{\frac{2}{\gamma}}\right), \quad c_1 = \frac{3a_1\alpha_0}{2} = \frac{-\alpha_0}{\left(1 \pm \sqrt{\frac{2}{\gamma}}\right)}, \quad C_2 = \frac{c_1}{c_0} = \frac{2\gamma\alpha_0}{3\gamma - 6}, \quad \gamma \neq 2. \tag{87}$$

The parametric form of the time variable is given by the equation

$$t(\eta) = \int y^{-a} d\eta. \tag{88}$$

Therefore, the main dynamical variables are given in parametric form as follows

$$
\begin{aligned}
H &= y^a, \\
q &= y^a \frac{d}{d\eta}\left(\frac{1}{H}\right) - 1, \\
\Lambda &= 3\beta H^2 = 3\beta y^{2a}, \qquad \beta = 1 - 2/n, \\
G &= bH^{-n\beta} = by^{-2\gamma\beta}, \\
\rho &= \rho_0 H^n = \rho_0 y^{2\gamma}, \\
\Pi &= \frac{-\left(y^a H' + \frac{3}{2}H^2\right)}{4\pi G}, \\
T &= T_0 \rho^{\frac{\gamma-1}{\gamma}} = T_0 y^{2(\gamma-1)}, \\
f &= \exp\left(\eta\right), \\
\Sigma &= -3 \int \Pi f^3 H T^{-1} \left(y^a\right)^{-1} d\eta.
\end{aligned}
\tag{89}
$$

As we can see, the particular solution given by Eq. (86) is quite similar to the one obtained in Case 1 (see Eq. (48)), except than here, the value of the constant parameters m, n, etc, take different values. For this reason, the particular solution of the FE given by Eq. (89), shows a very similar behavior to the one provided in Eq. (52) which has been previously discussed.

2. Solution for $\gamma = 2$

By assuming $\gamma = 2$ and $s = 1/2$, the following parameters are obtained: $a_{1+} = -1/3$, $\alpha_1 = 0$, $n = 4$ and $\beta = 1/2$. Therefore, Eq. (80) simplifies as

$$
y' + \frac{\alpha_0}{2}y^2 = k_1 \mathrm{e}^{-3\eta},
\tag{90}
$$

with solution

$$
y(\eta) = \sqrt{\frac{2k_1}{\alpha_0}}\,\mathrm{e}^{-3\eta/2}\left(\frac{2Y_1(\frac{1}{3}\sqrt{2k_1\alpha_0}\mathrm{e}^{-3\eta/2}) + k_2 J_1(\frac{1}{3}\sqrt{2k_1\alpha_0}\mathrm{e}^{-3\eta/2})}{2Y_0(\frac{1}{3}\sqrt{2k_1\alpha_0}\mathrm{e}^{-3\eta/2}) + k_2 J_0(\frac{1}{3}\sqrt{2k_1\alpha_0}\mathrm{e}^{-3\eta/2})}\right),
\tag{91}
$$

where Y_1 is the Bessel function of second kind and order 1, J_1 is the Bessel function of first kind and order 1, and k_2 is an integration constant.

Therefore, according to Eq. (24)

$$
t(\eta) = \int \left(\sqrt{\frac{2k_1}{\alpha_0}}\,\mathrm{e}^{-3\eta/2}\left(\frac{2Y_1(\frac{1}{3}\sqrt{2k_1\alpha_0}\mathrm{e}^{-3\eta/2}) + k_2 J_1(\frac{1}{3}\sqrt{2k_1\alpha_0}\mathrm{e}^{-3\eta/2})}{2Y_0(\frac{1}{3}\sqrt{2k_1\alpha_0}\mathrm{e}^{-3\eta/2}) + k_2 J_0(\frac{1}{3}\sqrt{2k_1\alpha_0}\mathrm{e}^{-3\eta/2})}\right)\right)^{-1} d\eta.
\tag{92}
$$

The main dynamical variables of the FE for this case are given in parametric form as follows

$$
\begin{aligned}
f &= f_0 \exp\left(\eta\right), \\
H &= y, \\
q &= y\frac{d}{d\eta}\left(\frac{1}{y}\right) - 1, \\
\rho &= \rho_0 y^4, \\
\Pi &= -\frac{1}{4\pi G}\left(yH' + \frac{3}{2}H^2\right), \\
T &= T_0 \rho^{1/2} = T_0 y^2, \\
\Sigma &= -\frac{3}{k_B}\int \Pi f^3 H T^{-1} y^{-1} d\eta, \\
G &= by^{-4\beta}, \\
\Lambda &= 3\beta y^2.
\end{aligned}
\tag{93}
$$

We have plotted the solution (93) in Figs. (5) and (6) in order to analyze the dynamical behavior. The energy density of the cosmological fluid, represented in Fig. (5) is a decreasing function of time. The bulk viscous pressure, shown in Fig. (5), satisfies the condition $\Pi < 0$, only for time intervals greater than an initial value t_*. Hence, this model can describe the dynamics of the causal bulk viscous Universe with variable gravitational and cosmological constants only for a finite time interval when $t > t_*$. Furthermore, the solution is thermodynamically consistent since the relationship $|\Pi|/p \ll 1$ is verified. The evolution of the Universe starts in a non-expansionary phase, with the deceleration parameter $q > 0$, represented in Fig. (5), but it enters into an inflationary phase quickly, so the Universe ends in an inflationary epoch.

The behavior of the comoving entropy has been plotted in Fig. (6). As we can see, this quantity is growing so the solution is thermodynamically consistent. The time variation of the cosmological and gravitational constants is represented in Fig. (6). The gravitational constant G is always a time growing function, while the cosmological constant is a positive decreasing function of time.
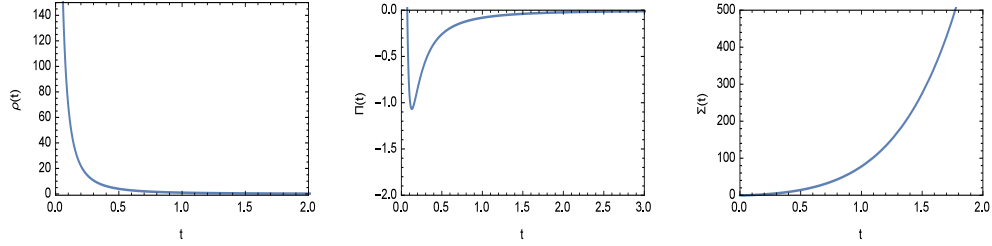

FIG. 5: Plots of the solution (93). Energy density $\rho(t)$, bulk viscous pressure $\Pi(t)$, and the entropy $\Sigma(t)$, for $\gamma = 2$, $s = 1/2$, $k_1 = 1$, $k_2 = -1$, and $\alpha_0 = 1$.
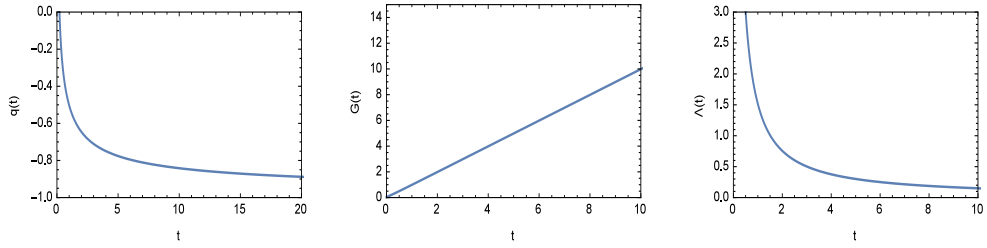

FIG. 6: Plots of solution (93). Deceleration parameter $q(t)$, $G(t)$ and $\Lambda(t)$. Parameter values as in Fig. 5

IV. FACTORIZATION METHOD FOR APPROACH 2

In this Section, we study the Eq. (23) which is a nonlinear second order differential equation in ρ which depends on an unknown function $G(t)$. Therefore, in order to apply the standard procedure of the factorization method, in Scheme 1, we impose the mathematical assumption $G = G_0\rho^n$ to obtain an ODE depending only on ρ. In order to avoid such assumption, in Scheme 2, we develop a new factorization scheme that allows to construct the explicit form of the time depending gravitational constant G. In both cases, we have been able to obtain particular solutions of the monomial type $G(t) \sim t^\varepsilon$.

A. Scheme 1

In this first scheme, we consider the assumption $G = G_0\rho^n$, then Eq. (23) reads

$$\ddot{\rho} - \frac{1}{\rho}\dot{\rho}^2 + D\rho^{1-s}\dot{\rho} + B\rho^{n+2} = 0, \tag{94}$$

where $D = \frac{2(1-s)\gamma}{\alpha}$, and $B = -G_0 A = -\frac{12\pi G_0(\gamma-\chi)(\chi-2)\gamma}{\chi} > 0$. Note that the case $s = 1$ must be studied separately in order to avoid $D = 0$.

Assuming the factorization scheme provided in Eqs. (28)-(34), with independent variable replaced by the time variable, the Eq. (94) admits the factorization

$$\left[D_t - \frac{1}{\rho}\dot{\rho} - a_1\sqrt{B}\rho^{\frac{n+1}{2}} \right] \left[D_t - a_1^{-1}\sqrt{B}\rho^{\frac{n+1}{2}} \right] \rho = 0, \tag{95}$$

where the following functions ϕ_i,

$$\phi_1 = \frac{1}{\rho}, \quad \phi_2 = a_1\sqrt{B}\rho^{\frac{n+1}{2}}, \quad \text{and} \quad \phi_3 = a_1^{-1}\sqrt{B}\rho^{\frac{n+1}{2}}, \tag{96}$$

with $a_1(\neq 0)$ a real arbitrary constant, have been introduced.

The explicit form of the constant a_1 is obtained by substituting the functions ϕ_i into Eq. (31). Then, we get

$$\left(-a_1\sqrt{B} - (1-s)a_1^{-1}\sqrt{B} \right)\rho^{\frac{n+1}{2}} = D\rho^{1-s}, \tag{97}$$

and thus

$$a_{1\pm} = -\frac{1}{2\sqrt{B}}\left(D \mp \sqrt{4B(s-1) + D^2} \right). \tag{98}$$

Also, the relationship $n = 1 - 2s$ should be satisfied.

For the chosen factorizing functions (96), the Eq. (34) of the factorization scheme provides the following result for the function $\Omega = \Omega(\rho, t)$,

$$\Omega = \kappa_1 \rho e^{a_1\sqrt{B}\int \rho^{\frac{n+1}{2}} dt}, \tag{99}$$

where κ_1 is an integration constant. Then, Eq. (95) turns into the equation

$$\dot{\rho} - \frac{\sqrt{B}}{a_1}\rho^{\frac{n+3}{2}} = \kappa_1 e^{a_1\sqrt{B}\int \rho^{\frac{n+1}{2}} dt}\rho, \tag{100}$$

whose solution is also solution of Eq. (94). The general solution of Eq. (100) is obtained in parametric form by performing the following transformation of the independent variable

$$d\nu = \rho^{\frac{n+1}{2}} dt, \tag{101}$$

which leads to the differential equation

$$\frac{d\rho}{d\nu} - \frac{\sqrt{B}}{a_1}\rho = \kappa_1 e^{a_1\sqrt{B}\nu}\rho^{\frac{1-n}{2}}, \tag{102}$$

with general solution given in the form

$$\rho(\nu) = \left[\frac{a_1\kappa_1(1+n)}{\sqrt{B}(2a_1^2 - 1 - n)}e^{\sqrt{B}a_1\nu} + \kappa_2 e^{\frac{\sqrt{B}(1+n)\nu}{2a_1}} \right]^{\frac{2}{1+n}}, \tag{103}$$

where κ_2 is an integration constant. The parametric form of the time variable is obtained via the equation

$$t(\nu) = \int \rho^{-\frac{n+1}{2}}(\nu)d\nu. \tag{104}$$

The main FE variables are given in parametric form as follows

$$\rho = \left[\frac{a_1 \kappa_1 (1-s)}{\sqrt{B}(a_1^2 + s - 1)} e^{\sqrt{B} a_1 \nu} + \kappa_2 e^{\frac{\sqrt{B}(1-s)\nu}{a_1}} \right]^{\frac{1}{1-s}},$$

$$G = G_0 \rho^{1-2s},$$

$$\Pi = -\chi \rho,$$

$$H = H_0 \rho^{1-s},$$

$$f = e^{H_0 \nu}, \tag{105}$$

$$q = \rho^{1-s} \frac{d}{d\nu}\left(\frac{1}{H}\right) - 1,$$

$$\Lambda = -8\pi \int \frac{dG}{d\nu} \rho \, d\nu,$$

$$T = T_0 \rho^{1-\frac{1}{\gamma}},$$

$$\Sigma = -\frac{3}{k_B} \int^\nu \frac{\Pi f^3 H}{T \rho^{\frac{n+1}{2}}} d\nu.$$

The solution is new and it is valid for $s \in [0,1)$, that is, $s \neq 1$ and $\gamma \in [1,2]$, under the assumption $G = G_0 \rho^n$. In order to understand the solution (105), we have plotted the behavior of the main quantities in Figs. (7) and (8) by setting the following values for the constant parameters: $\gamma = 4/3$, $\chi = .3$, $\alpha = .038$, $G_0 = 1$, $\kappa_1 = 100$, $\kappa_2 = .1$, and $s = .4$ (dashed line), $s = .5$ (solid line), $s = .7$ (dotted line).

In Fig. (7) we can see that the energy density of the cosmological fluid is a decreasing function of time in all the studied cases. The bulk viscous pressure satisfies the condition $\Pi < 0$, thus this solution may describe the dynamics of the causal bulk viscous Universe with variable gravitational and cosmological constants. In this solution, with the fixed numerical values, the ratio of the bulk viscous pressure and the thermodynamical pressure is lesser than one, that is, $|\Pi|/p = .9 < 1$. Consequently, during this period the model is consistent thermodynamically. The behavior of the comoving entropy has been plotted in Fig. (7). As we can see, this quantity is always growing, in fact it grows quickly in all the cases.

The evolution of the Universe, represented by the deceleration parameter $q(t)$ in Fig. (8), starts in a non-accelerating phase and ends in an inflationary phase ($q < 0$) in two cases. The time variation of the cosmological and gravitational constants is also shown in Fig. (8). The gravitational constant G behaves as a growing time function if $s > 1/2$, it is constant, $G = G_0$, for $s = 1/2$, and decreases for $s < 1/2$. The cosmological constant Λ behaves like a decreasing time function, but its sign depends on the value of s in such a way that it is positive for $s > 1/2$, vanishes for $s = 1/2$ and it is negative for $s > 1/2$.
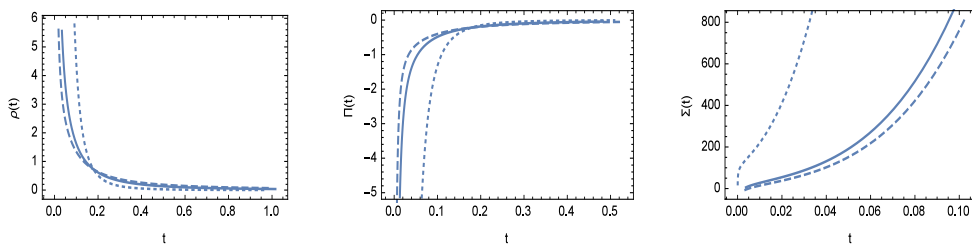

FIG. 7: Plots of solution (105). Energy density $\rho(t)$, bulk viscous pressure $\Pi(t)$, and the entropy $\Sigma(t)$, for $\gamma = 4/3$, $\chi = .3$, $\alpha = .038$, $G_0 = 1$, $\kappa_1 = 100$, $\kappa_2 = .1$, and $s = 0.4$ (dashed line), $s = 0.5$ (solid line), $s = 0.7$ (dotted line).

Furthermore, the following particular solution (power-law solution) of Eq. (100) is obtained for $\kappa_1 = 0$,

$$\rho(t) = 4^{\frac{1}{2(1-s)}} \left[-(1+n)\left(\frac{\sqrt{B}}{a_1} t + \kappa_2 \right) \right]^{-\frac{1}{(1-s)}}, \tag{106}$$
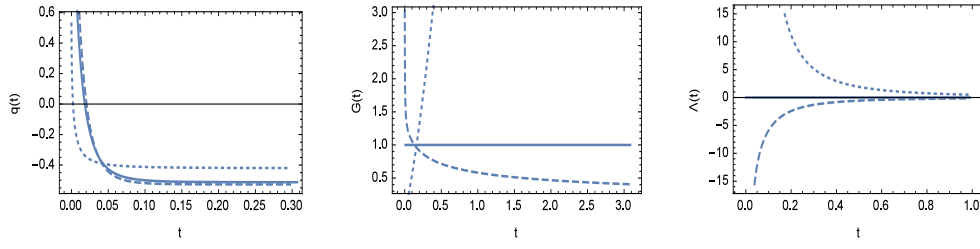
where κ_2 is an integration constant.

FIG. 8: Plots of solution (105). Deceleration parameter $q(t)$, $G(t)$ and $\Lambda(t)$. Parameter values as in Fig. 7.

The main FE quantities are given for $\kappa_2 = 0$ as follows

$$\rho(t) = \rho_0 t^{\frac{1}{s-1}}, \qquad \rho_0 = 4^{\frac{1}{2(1-s)}} \left(-2(1-s)\frac{\sqrt{B}}{a_1} \right)^{-\frac{1}{(1-s)}},$$

$$G(t) = G_0 \rho^{1-2s}(t) = G_1 t^{\frac{1-2s}{s-1}}, \quad G_1 = G_0 \rho_0^{1-2s},$$
$$\Pi(t) = -\chi \rho(t),$$

$$H(t) = H_0 \rho^{(1-s)}(t) = H_0 t^{-1}, \qquad H_0 = \left(\frac{3\rho_0^{2(1-s)}}{4\pi(1-s)} \right)^{1/2},$$

$$f(t) = f_0 t^{H_0},$$
$$q(t) = \frac{d}{dt}\left(\frac{1}{H(t)} \right) - 1 = \frac{1}{H_0} - 1,$$
$$\Lambda(t) = -8\pi \int \dot{G}(t)\rho(t)dt = \Lambda_0 t^{-2},$$

$$\Sigma(t) = -\frac{3}{k_B} \int_{t_0}^{t} \frac{\Pi(t)f^3(t)H(t)}{T(t)}dt = \frac{-3\Sigma_0(n+1)}{-\frac{2}{\gamma} + 3H_0(1+n)} t^{\frac{1}{n+1}\left(-\frac{2}{\gamma} + 3H_0(1+n)\right)},$$

(107)

where $\Sigma_0 = \left(-\chi f_0^3 H_0 T_0^{-1}\right) < 0$, and $s \neq 1$. Also, in order to obtain firm results from the thermodynamical point of view we can consider the condition

$$\left| \frac{\Pi}{p} \right| \ll 1, \qquad \chi < \gamma - 1,$$

(108)
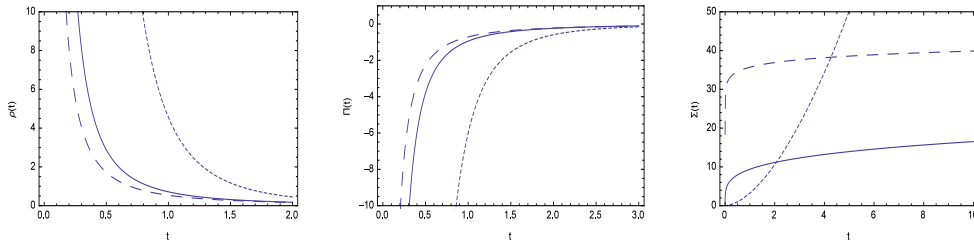
although it is not strictly necessary.

FIG. 9: Plots of solution (107). Energy density $\rho(t)$, bulk viscous pressure $\Pi(t)$, and the entropy $\Sigma(t)$, for $\gamma = 4/3$, $\chi = 1.32$, $\alpha = 1$, $G_0 = 1$, $\kappa_2 = 0$, and $s = 0.4$ (dashed line), $s = 0.5$ (solid line), $s = 0.7$ (dotted line).

The solution is valid for $s \in [0, 1)$ and $\gamma \in [1, 2]$. In order to understand the solution (107), we have plotted the behavior of the main quantities in Figs. (9) and (10) by setting the following values for the constant parameters: $\gamma = 4/3$, $\chi = 1.32$, $\alpha = 1$, $G_0 = 1$, $\kappa_1 = 1$, $\kappa_2 = 1$, and $s = 0.4$ (dashed line), $s = 0.5$ (solid line), $s = 0.7$ (dotted line). As we can see in Fig. (9), the energy density of the cosmological fluid is a decreasing function of time in all
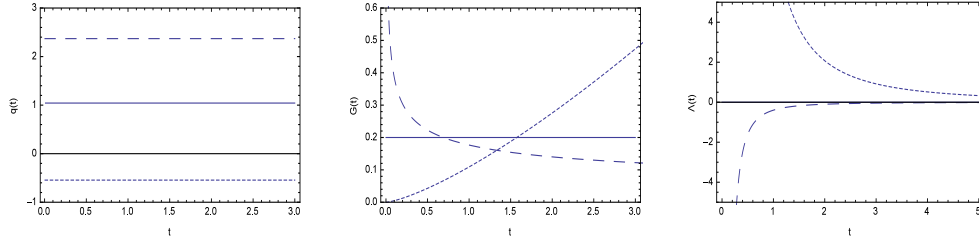
FIG. 10: Plots of solution (107). Deceleration parameter $q(t)$, $G(t)$ and $\Lambda(t)$. Parameter values as in Fig. 9

the studied cases. Also, the bulk viscous pressure satisfies the condition $\Pi < 0$, thus this solution may describe the dynamics of the causal bulk viscous Universe with variable gravitational and cosmological constants. But in this solution, for the fixed numerical values, the ratio of the bulk viscous pressure and the thermodynamical pressure is greater than one, that is, $|\Pi|/p = \chi/(\gamma - 1) \gg 1$, since we have set $\chi = 1.32$. Consequently, the solution is not consistent thermodynamically. The behavior of the comoving entropy has been plotted in Fig. (9). As it can be seen, this quantity is always growing, but it grows faster in the case $s = 1/2$.

The evolution of the Universe is noninflationary, with deceleration parameter $q > 0$, for the cases $s \leq 1/2$, although it is inflationary for $s = 0.7$. This behavior is shown in Fig. (9). The time variation of the cosmological and gravitational constants is also shown in Fig. (10). The gravitational constant G is a growing time function for $s > 1/2$, constant for $s = 1/2$, and decreasing for $s < 1/2$. The cosmological constant is a positive decreasing function of time if $s > 1/2$, vanishes for $s = 1/2$ ($G = G_0$), and it is negative for $s < 1/2$. In this solution, we have obtained, $G(t) \sim t^\varepsilon$, thus this means that $\Delta G = G'/G \sim \varepsilon t_0^{-1}$, with $\varepsilon = (1 - 2s)/(s - 1)$. By taking an average estimation for the age of the universe as $t_0 = 1.3798 \pm 0.037 \times 10^{10}$yr [56], we obtain different results for different values of $s \in [0, 1)$. Note that ΔG depends on the value of the estimation of the Hubble parameter. Therefore, -7.24742×10^{-11}yr$^{-1} \leq \Delta G \leq 5.79794 \times 10^{-10}$yr^{-1}; for example, for $s = 0.6$, $\Delta G = 3.62371 \times 10^{-11}$yr^{-1}, which is in agreement with [57] and [58], while for $s = 0.4$, $\Delta G = -2.4158 \times 10^{-11}$yr^{-1}, which is in agreement with [59].

The main difference between the particular solution given by Eq. (107) and the general solution given by Eq. (105) is reflected in the behavior of the quantities entropy and deceleration parameter $q(t)$. In the particular solution the entropy grows slowly, while in the general solution it grows quickly in all the studied cases. Regarding the deceleration parameter, we have shown that for some choices of numerical values of the parameters, the particular solution may behave as not inflationary, while in the general case (see Fig. (10)) all the solutions end in an accelerating era.

B.   Scheme 2

A very interesting scheme, in this second approach, is developed by studying the factorization of the variable coefficient nonlinear ODE for the energy density

$$\ddot{\rho} - \frac{1}{\rho}\dot{\rho}^2 + D\rho^{1-s}\dot{\rho} - AG(\rho, t)\rho^2 = 0, \tag{109}$$

with the constant coefficients $D = \frac{2(1-s)\gamma}{\alpha}$ and $A = \frac{12\pi(\gamma - \chi)(\chi - 2)\gamma}{\chi} < 0$. Since we assume $\rho = \rho(t)$, the gravitational constant is conveniently written as $G = G(\rho, t)$, which is indirectly a function of t.

The factorization method allows to get a complete solution by developing an explicit form of the arbitrary function $G = G(\rho, t)$. Let us consider the following factorization in differential operators applied to $\rho(t)$

$$[D_t - \phi_1(\rho, t)\dot{\rho}][D_t - \phi_2(\rho, t)]\rho = 0, \tag{110}$$

where $D_t = \dot{} = \frac{d}{dt}$. The operators are developed in Eq. (110) to find the equation

$$\ddot{\rho} - \phi_1\dot{\rho}^2 - \left(\frac{\partial\phi_2}{\partial\rho}\rho + \phi_2 - \phi_1\phi_2\rho\right)\dot{\rho} - \frac{\partial\phi_2}{\partial t}\rho = 0. \tag{111}$$

Therefore, in order to factorize Eq. (109) in the form (110), the following restriction equations are obtained by

comparing Eqs. (109) and (111),

$$\phi_1 = \frac{1}{\rho}, \tag{112}$$

$$\phi_2 - \phi_1\phi_2\rho + \frac{\partial \phi_2}{\partial \rho}\rho = -D\rho^{1-s}, \tag{113}$$

$$\frac{\partial \phi_2}{\partial t} = AG(\rho, t)\rho. \tag{114}$$

Eqs. (112) and (113) generate the function

$$\phi_2(\rho, t) = \frac{D}{s-1}\rho^{1-s} + T(t), \tag{115}$$

then, from Eq. (114) we get

$$\frac{\partial \phi_2}{\partial t} = \dot{T}(t) = AG(\rho, t)\rho, \tag{116}$$

which implies

$$G(\rho, t) = G_1(t)\rho^{-1}, \tag{117}$$

where $G_1(t)$ is an arbitrary function of t, and

$$T(t) = A \int G_1(t)dt. \tag{118}$$

Assuming the equation

$$[D_t - \phi_2(\rho, t)]\rho = \Omega(\rho, t), \tag{119}$$

then, from Eq. (110) we get

$$\dot{\Omega} - \frac{\dot{\rho}}{\rho}\Omega = 0, \tag{120}$$

with solution $\Omega = k_0\rho$, where k_0 is an integration constant. Therefore, from Eq. (119) we get the nonlinear first order ODE

$$\dot{\rho} - \frac{D}{s-1}\rho^{1-s} + \left(A \int G_1(t)dt\right)\rho = k_0\rho. \tag{121}$$

The general solution of Eq. (121) is also solution of Eq. (109), and it is given by

$$\rho(t) = e^{\zeta_1}\left(D \int e^{\zeta_2}dt + k_1\right)^{(1/(s-1))}, \tag{122}$$

where

$$\zeta_1 = A \int \int G_1(t)dtdt + k_0 t \tag{123}$$

$$\zeta_2 = (s-1)\int\left(-A \int G_1(t)dt - k_0\right)dt, \tag{124}$$

and k_1 is an integration constant.

We can obtain solutions for different choices of the function $G_1(t)$ and the integration constants. As an example, we choose

$$G_1(t) = \frac{1}{t^2}, \qquad k_0 = 0, \tag{125}$$

such that

$$A \int \int G_1(t) dt dt = -A \ln t, \tag{126}$$

then, we get

$$\rho(t) = t^{-A} \left(D \int \frac{1}{t^{-A(s-1)}} dt + k_1 \right)^{(1/(s-1))}, \tag{127}$$

and by setting $k_1 = 0$,

$$\rho(t) = \rho_0 t^{1/(s-1)}, \qquad \text{where} \quad \rho_0 = \left(\frac{D}{A(s-1)+1} \right)^{(1/(s-1))}, \tag{128}$$

which is equivalent to the solution (107) obtained in the previous section and discussed in [42].

V. CONCLUSIONS.

We have studied a full causal bulk viscous cosmological model with time varying constants $G(t)$ and $\Lambda(t)$. The geometry of the universe is described by a spatially flat FRW metric. In order to derive the FE we have assumed that the Bianchi identity is verified (see Eq. (4)), as well as the matter conservation described by Eq. (6). We have studied two phenomenological approaches that allow us to obtain nonlinear second order ODE, Eq. (18) and Eq. (23), which describe the dynamics of the cosmological model. We would like to emphasize the fact that one of the advantages of the factorization method, as opposed to different approaches followed by other authors, is that it allows us to obtain relationships between the free parameters entering the ODE (γ, s, n, etc). Another advantage is that it is a well established method to find exact solutions through algebraic procedures to nonlinear ODE. In this way, we have been able to find new parametric exact solutions of the FE. Furthermore, a new factorization scheme to generate exact solutions of a kind of nonlinear second order ODE with variable coefficients has been developed in Section IV B. These results have not been previously reported. Further results are currently under study.

For the first approach, developed in Section III, we have obtained the following solutions of the FE by departing from the Eq. (18) for the Hubble function $H(t)$. In Case 1, which is the most general case, the solution is valid for $\gamma \in [1,2)$ and $s \in [0,1]$ (see Eq. (52)), and it predicts that G and Λ vary as functions of the time, but in the long time limit they tend to a constant value, that is $G = G_0$, and Λ takes values very close to zero. The mathematical solutions do not allow us to know if G is growing or decreasing, both cases are allowed (they are thermodynamically consistent, with ρ decreasing, $\Pi < 0$, etc.) and they are in agreement with the observations, since some of them conclude that G is growing while another ones indicate that G is decreasing (see for instance [28] or for a short review [60]). We take into account some observational data which indicate us that $\Lambda > 0$ (see [61, 62]). Then, we consider those solutions in agreement with these observations ($\Lambda > 0$), and conclude that G is growing in the very early universe, since the behavior of Λ and G are related by Eq. (6) derived from the Bianchi identity; we conclude that if G is growing then $\Lambda > 0$, if G is decreasing then $\Lambda < 0$, and if G behaves as a true constant then Λ vanishes. Therefore, in another theoretical framework, as for example the JBD model, it is possible to arrive at different conclusions (see for example [63]). Since this solution is not valid for $\gamma = 2$, in Case 2 we have studied the solution for such a specific and important case, the ultra stiff matter. In this case (see Eq. (70)), the solution is valid for $\gamma = 2$ and $s \in [0, 3/4)$. Nevertheless, the behavior of G and Λ are completely different from the first case: G is always growing ($\Lambda > 0$) or decreasing ($\Lambda < 0$), but they do not tend to a constant value. Only in the case of $s = 1/4$ we find that $G = G_0$ and therefore $\Lambda = 0$. In Case 3, we have studied some particular cases of the original equation (25). Under the assumption $a = 2\gamma/n$, we make $A_1 = 0$, so Eq. (25) considerably simplifies. Under this hypothesis we have found three solutions. In the first of them, the solution given by Eq. (89) is quite similar (but not identical) to the one given in Eq. (52), and therefore, we arrive to similar conclusions. In the second solution given by Eq. (93), we set $\gamma = 2$ and $s = 1/2$. In this case, G is always growing while Λ behaves as a positive time decreasing function. To the best of our knowledge these solutions have not been previously reported in the literature.

In the second approach, developed in Section IV, we have studied Eq. (23) for the energy density $\rho(t)$. We have obtained two solutions. The first of them is provided in Eq. (105) and is valid for $\gamma \in [1,2]$ and $s \in [0,1)$, that is, $s \neq 1$. It is a new general solution of Eq. (23). We find that G could be growing ($\Lambda > 0$) as well as decreasing ($\Lambda < 0$), and they behave as constants ($G = G_0$, $\Lambda = 0$) only in the particular case where $s = 1/2$. The second solution, provided in Eq. (107), is a particular solution of Eq. (23), and it has been previously reported in the literature in Ref. [42]. It shows a similar behavior to solution (105) with regard to G and Λ, but not with regard to the deceleration parameter

with non-inflationary solutions. In Section IV B, we have ended our study of this cosmological model by presenting a new factorization scheme, which allows us to generate new exact solutions to the nonlinear second order ODE with variable coefficients (109) for the energy density $\rho(t)$. As a specific case, a power law solution previously reported in the literature [42] has been obtained.

Acknowledgements

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