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Gamma-Maxwell distribution

Yuri A. Iriarte* Juan M. Astorga † Heleno Bolfarine ‡
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Abstract

A new two-parameter distribution, the gamma-Maxwell distribution, is proposed and studied. We generate the new distribution using the gamma-G generator of distributions proposed by Zografos and Balakrishnan (2009). The proposal distribution can be seen as an extension of the Maxwell distribution with more flexibility in terms of the distribution asymmetry and kurtosis. We study some probability properties, discuss maximum likelihood estimation and present a real data application indicating that the new distribution can improve the ordinary Maxwell distribution in fitting real data.

KEY WORDS: Maxwell distribution, Gamma-G generator, Gamma-Maxwell distribution, Maximum Likelihood.

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1 Introduction

A random variable X follows a Maxwell distribution with scale parameter θ, denoted $X \sim M(\theta)$, if its probability density function (pdf) and cumulative distribution function (cdf) are given by

$$g(x;\theta) = \frac{4}{\sqrt{\pi}}\theta^{3/2}x^2 e^{-\theta x^2}, \tag{1}$$

and

$$G(x;\theta) = \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \theta x^2\right), \tag{2}$$

respectively, where $x > 0$ and $\gamma(a,x) = \int_0^x u^{a-1}e^{-u}\,du$ is the incomplete gamma function. Tyagi and Bhattacharya (1989a, 1989b) obtained the minimum variance unbiased estimator, Bayes estimator and the reliability function of this distribution. Chaturvedi and Rani (1998) generalized the Maxwell distribution and obtained classical and Bayesian estimators for this distribution. Bekker and Roux (2005) studied empirical Bayes estimation for Maxwell distribution. Shakil et al. (2008) studied the distributions of the product $|XY|$ and ratio $|X/Y|$, when X and Y are independent random variables having the Maxwell and Rayleigh distributions. Kazmi et al. (2012) obtained the Bayesian estimation for two component mixture of Maxwell distribution assuming type I censored data.

Zografos and Balakrishnan (2009) proposed families of univariate distribution generated by gamma random variables. For any baseline cdf $G(x)$, $x \in \mathbb{R}$, they defined the gamma-G generator (with an extra shape parameter $\alpha > 0$) by the pdf and cdf given by

$$f(x) = \frac{g(x)}{\Gamma(\alpha)}\{-\log[1 - G(x)]\}^{\alpha-1}, \tag{3}$$

and

$$F(x) = \frac{\gamma(\alpha, -\log[1 - G(x)])}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)}\int_0^{-\log[1-G(x)]} u^{\alpha-1}e^{-u}\,du, \tag{4}$$

respectively, where $g(x) = dG(x)/dx$, $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1}e^{-u}\,du$ is the gamma function and $\gamma(\cdot)$ is the incomplete gamma function.

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In this article, we introduce a new univariate distribution using the gamma-G generator of distributions. Specifically, we replace (1) and (2) into (3) to obtain the pdf of the new distribution. The respective cdf is obtained replacing (2) into (4).

The article is organized as follows. In Section 2 we present its density, moments and asymmetry and kurtosis coefficients. In Section 3 we discuss moment and maximum likelihood estimations. In addition, we calculate the elements of the observed information matrix and conduct a simulation study to illustrate the behavior of the maximum likelihood estimates. Section 4 present two application to real data sets. The applications illustrates the good performance of the model proposed in real applications. Final conclusions are reported in Section 5.

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2 Gamma-Maxwell distribution

In this section, we present the pdf and cdf of the new distribution. In addition, we derive an analytical expression for distributional moments and use this result to compute the asymmetry and kurtosis coefficients.

Definition 2.1 *A random variable X follows a Gamma-Maxwell (GM) distribution, denoted as $X \sim GM(\theta, \alpha)$, if its probability density function (pdf) and cumulative distribution function (cdf) are given by*

$$f(x;\theta,\alpha) = \frac{4\theta^{3/2}}{\sqrt{\pi}\Gamma(\alpha)} x^2 e^{-\theta x^2} \left\{ -\log\left[1 - \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \theta x^2\right)\right]\right\}^{\alpha-1}, \tag{5}$$

and

$$F(x;\alpha,\theta) = \frac{\gamma\left(\alpha, -\log\left(1 - \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \theta x^2\right)\right)\right)}{\Gamma(\alpha)}, \tag{6}$$

respectively, where $x > 0$, $\theta > 0$ is a scale parameter, $\alpha > 0$ is a shape parameter, $\Gamma(\cdot)$ is the gamma function and $\gamma(\cdot)$ is the incomplete gamma function.

If $\alpha = 1$ the gamma-Maxwell distribution is reduced to the Maxwell distribution. Figure 1 depicts some of the shapes that the gamma-Maxwell distribution can take for different values of its parameters.

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Next, we present some transformations related to GM distributions.

Proposition 2.1 *Let* $X \sim GM(\theta, \alpha)$*. Then,*

a) $W = aX \sim GM(\theta/a^2, \alpha)$ *for all* $a > 0$*;*

b) The pdf of $W = X^{-1}$ *is*

$$f_W(w; \theta, \alpha) = \frac{4\theta^{3/2}}{\sqrt{\pi}\Gamma(\alpha)} w^{-4} e^{-\frac{\theta}{w^2}} \left\{ -\log\left[1 - \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{\theta}{w^2}\right)\right]\right\}^{\alpha-1}, \; w > 0;$$

c) The pdf of $W = \log(X)$ *is given by*

$$f_W(w; \theta, \alpha) = \frac{4\theta^{3/2}}{\sqrt{\pi}\Gamma(\alpha)} e^{2w - \theta e^{2w}} \left\{ -\log\left[1 - \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \theta e^{2w}\right)\right]\right\}, \; w \in \mathbb{R}.$$

Proof. Parts *a*)-*c*) are directly obtained from the change-of-variable method □.

Remark 1 *Part a) of Proposition 2.1 indicates that the GM distributions belong to the scale family, Part b) demonstrates that these distributions are not closed under reciprocation, while the result in Part c) can be used to study regression models in same lines as in the context of regression models for positive random variables; see McDonald and Butler(1990). In addition, Part a) allows us to obtain a one parameter GM distribution. That is, if* $X \sim GM(\theta, \alpha)$*, then* $\sqrt{\theta}X \sim GM(1, \alpha)$*.*

2.1 Moment and related measures

Proposition 2.2 *Let* $X \sim GM(\theta, \alpha)$*. Then, for* $r = 1, 2, ...$ *it follows that r-th moment is given by*

$$\mu_r = E(X^r) = \frac{\theta^{r/2}}{\Gamma(\alpha)} a_r \;, \tag{7}$$

where a_r *is defined as*

$$a_r = \int_0^1 \left(H^{-1}\left(u; \frac{3}{2}, 1\right)\right)^{\frac{r}{2}} \{-\log(1-u)\}^{\alpha-1} \; du, \tag{8}$$

where H^{-1} *is the quantile function of the gamma distribution.*

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Proof. The cdf of the Maxwell distribution that is shown in Equation (2), can be written as

$$G(x,\alpha) = \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\theta x^2\right) = H\left(\theta x^2;\frac{3}{2},1\right),$$

where $H(x;a,b) = \int_0^x \frac{b^a}{\Gamma(a)}u^{a-1}e^{-bx}\,du$ is the cdf of the gamma distribution.
Using the defining moments, the r-th moment is given by

$$\mu_r = \frac{4\theta^{3/2}}{\sqrt{\pi}\Gamma(\alpha)}\int_0^\infty x^{r+2}e^{-\theta x^2}\left\{-\log\left[1-H\left(\theta x^2;\frac{3}{2},1\right)\right]\right\}^{\alpha-1}dx.$$

Now, by letting $u = H\left(\theta x^2,\frac{3}{2},1\right)$ and considering a_r as the integral, the result is obteined □.

Corollary 2.1 *If $X \sim GM(\theta,\alpha)$, then*

$$E(X) = \frac{a_1}{\sqrt{\theta}\Gamma(\alpha)} \qquad \text{and} \qquad Var(X) = \frac{\Gamma(\alpha)a_2 - a_1^2}{\theta\Gamma^2(\alpha)}.$$

Corollary 2.2 *If $X \sim GM(\theta,\alpha)$, then the coefficients of asymmetry ($\sqrt{\beta_1}$) and kurtosis (β_2) are, respectively,*

$$\sqrt{\beta_1} = \frac{\Gamma^2(\alpha)a_3 - 3\Gamma(\alpha)a_1a_2 + 3a_1^3}{\left[\Gamma(\alpha)a_2 - 2a_1^2\right]^{3/2}},$$

and

$$\beta_2 = \frac{\Gamma^3(\alpha)a_4 - 4\Gamma^2(\alpha)a_1a_3 + 6\Gamma(\alpha)a_1^2a_2 - 3a_1^4}{\left[\Gamma(\alpha)a_2 - a_1^2\right]^2}.$$

Remark 2 *If $\alpha = 1$ the asymmetry and kurtosis coefficients take the values 13.791 and 3.108, respectively, which correspond to those for the classical Maxwell distribution. Figures 2 and 3 depict plots for the asymmetry and kurtosis coefficients, respectively, of the Maxwell distribution and gamma-Maxwell distribution.*

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3 Inference

In this section we discuss moment and maximum likelihood estimations for the parameters θ and α of the Gamma-Maxwell distribution. In addition, we present the observed information matrix for the gamma-Maxwell distribution and conduct a simulation study to illustrate the behavior of maximum likelihood estimates.

3.1 Moment estimators

Proposition 3.1 *Let $X_1, \ldots, X_n$ a random sample for the random variable $X \sim GM(\theta, \alpha)$. Then, moment estimators for $\boldsymbol{\theta} = (\theta, \alpha)$ are given by*

$$\overline{X^2} a_1^2 - \overline{X}^2 \Gamma(\widehat{\alpha}_M) a_2 = 0 \qquad \text{and} \qquad \widehat{\theta}_M = \frac{a_1^2}{\overline{X}^2 \Gamma^2(\widehat{\alpha}_M)},$$

where $\overline{X}$ is the sample mean and $\overline{X^2}$ is the sample mean for square of the sample units.

Proof. Using (7), it follows that

$$E(X) = \frac{a_1}{\sqrt{\theta}\Gamma(\alpha)} \qquad \text{and} \qquad E(X^2) = \frac{a_2}{\theta\Gamma(\alpha)},$$

and replacing $E(X)$ by $\overline{X}$ and $E(X^2)$ by $\overline{X^2}$, we obtain a system of equations for which the solution leads to the moment estimators $(\widehat{\theta}_M, \widehat{\alpha}_M)$ for (θ, α) □.

The solution for the moment estimators can be obtained by using numerical procedures. For instance, to solve the right equation we can use the "uniroot" function built in R.

3.2 Maximum Likelihood estimation

For a random sample $X_1, \ldots, X_n$ from the distribution $GM(\theta, \alpha)$, the log likelihood function can be written as

$$l(\theta, \alpha) = n \log\left(\frac{4}{\sqrt{\pi}}\right) + \frac{3n}{2}\log(\theta) - n\log(\Gamma(\alpha)) + 2\sum_{i=1}^{n}\log(x_i) - \theta\sum_{i=1}^{n} x_i^2 \tag{9}$$

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$$+(\alpha-1)\sum_{i=1}^{n}\log\left\{-\log\left[1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\theta x^{2}\right)\right]\right\},$$

so that the maximum likelihood equations are given by

$$\frac{3n}{2\theta}-\sum_{i=1}^{n}x_i^2+2\sqrt{\frac{\theta}{\pi}}(\alpha-1)\sum_{i=1}^{n}\frac{x_i^3e^{-\theta x_i^2}}{\left\{\log\left[1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\theta x_i^2\right)\right]\right\}\left[1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\theta x_i^2\right)\right]}=0, \tag{10}$$

$$-n\Psi(\alpha)+\sum_{i=1}^{n}\log\left\{-\log\left[1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\theta x_i^{2}\right)\right]\right\}=0, \tag{11}$$

where Ψ is the digamma function. The maximum likelihood estimators for θ and α can be obtained solving the non-linear equations in (10-11). Those solutions can be obtained by using the function optim available in software R Development Core Team (2014), the specific method is the L-BFGS-B developed by Byrd et al. (1995) which allows box constraint. This uses a limited-memory modification of the quasi-Newton method.

Alternatively, based on the results in Ristic and Balakrishnan (2012), the MLE for θ and α can be obtained in the following way. First, we solve the equation(11) for α, obtaining

$$\alpha=\Psi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\log\left(-\log A(\theta)\right)\right), \tag{12}$$

where Ψ^{-1} is the inverse digamma function and $A(\theta)=1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\theta x_i^2\right)$. Replacing (12) in (10), we obtain

$$\frac{3n}{2\theta}-\sum_{i=1}^{n}x_i^2+2\sqrt{\frac{\theta}{\pi}}\left(\Psi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\log\left(-\log A(\theta)\right)\right)-1\right)\sum_{i=1}^{n}\frac{x_i^3e^{-\theta x_i^3}}{A(\theta)\log A(\theta)}=0.$$

Solving this non-linear equation for θ we obtain the MLE $\widehat{\theta}$. Finally, replacing $\widehat{\theta}$ in (12), we obtain the MLE $\widehat{\alpha}$.

3.3 Observed information matrix

For a random sample $X_1,\ldots,X_n$ from the distribution $GM(\theta,\alpha)$, so that the observed information matrix is given by

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$$I_n(\theta,\alpha)=\begin{pmatrix}\dfrac{\partial^2 l(\theta,\alpha)}{\partial\theta^2} & \dfrac{\partial^2 l(\theta,\alpha)}{\partial\alpha\partial\theta}\\ \dfrac{\partial^2 l(\theta,\alpha)}{\partial\theta\partial\alpha} & \dfrac{\partial^2 l(\theta,\alpha)}{\partial\alpha^2}\end{pmatrix},$$

such that

$$\begin{aligned}\frac{\partial^2 l(\theta,\alpha)}{\partial\theta^2} &= -\frac{3n}{2\theta^2}+\frac{(\alpha-1)}{\sqrt{\pi\theta}}\sum_{i=1}^{n}\frac{(1+2\theta x_i^2)x_i^3e^{-\theta x_i^2}}{\left\{\log\left[1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\theta x_i^2\right)\right]\right\}\left[1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\theta x_i^2\right)\right]}\\ &\quad+\frac{4\theta}{\pi}(\alpha-1)\sum_{i=1}^{n}\frac{x_i^6e^{-\theta x_i^2}\left\{1+\log\left[1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\theta x_i^2\right)\right]\right\}}{\left\{\log^2\left[1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\theta x_i^2\right)\right]\right\}\left[1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\theta x_i^2\right)\right]^2},\\ \frac{\partial^2 l(\theta,\alpha)}{\partial\alpha\partial\theta} &= 2\sqrt{\frac{\theta}{\pi}}\sum_{i=1}^{n}\frac{x_i^3e^{-\theta x_i^2}}{\left\{\log\left[1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\theta x_i^2\right)\right]\right\}\left[1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\theta x_i^2\right)\right]},\\ \frac{\partial^2 l(\theta,\alpha)}{\partial\theta\partial\alpha} &= 2\sqrt{\frac{\theta}{\pi}}\sum_{i=1}^{n}\frac{x_i^3e^{-\theta x_i^2}}{\left\{\log\left[1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\theta x_i^2\right)\right]\right\}\left[1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\theta x_i^2\right)\right]},\\ \frac{\partial^2 l(\theta,\alpha)}{\partial\alpha^2} &= -n\Psi_1(\alpha),\end{aligned}$$

where Ψ_1 is the trigamma function.

3.4 Simulation study

In this subsection, simulation is performed to illustrate the behavior of the MLE estimators and the moments estimators for parameters θ and α. We generate 1000 random samples of sizes $n = 50$, $n = 100$ and $n = 200$ from the distribution $GM(\theta,\alpha)$ for fixed values of the parameters. Random numbers $X \sim GM(\theta,\alpha)$ can be generated as

$$X=\left(\frac{1}{\theta}H^{-1}\left(1-e^{-H^{-1}(u;\alpha,1)};\frac{3}{2},1\right)\right)^{1/2},$$

where H^{-1} is the quantile function of the gamma distribution. Measures and empirical standard deviations are presented in Table 1. Here, the parameters are well estimated and there is clear

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indication that the estimates are asymptotically unbiased. Moreover, MLEs are more efficient than moment estimators, as expected. We used the moments estimators as the starting values to obtain the MLE.

4 Application

In this section we analyze two data sets using the Maxwell and gamma-Maxwell distributions. In addition, we analyze the data sets using the Weibull (W) and Gamma (G) distributions. These distributions are usually used for analyzing this type of data sets. The first data set corresponds to wind speed data, reported by Ahmad et al. (2009) , and the second data set correspond to energy consumption data, previously studied by Devore (2005).

4.1 Wind speed data

Ahmad et al. (2009) analyze the variations in wind speed on the east coast of peninsular Malaysia in order to create optimal conditions for designing wind turbines and wind farms. In the study they report 41323 observations associated to wind speed. Table (2) presents summary statistics for the wind speed data where b_1 and b_2 are the coefficients of asymmetry and kurtosis, respectively. Using results in Section 3.1, moment estimators were computed, leading to $\widehat{\theta} = 0.096$ and $\widehat{\alpha} = 0.447$. These estimates were then used as starting values for the optim algorithm for maximizing the likelihood function. Table 3 presents parameter estimates for the M, G, W and GM models, using maximum likelihood (MLE) approach and the corresponding Akaike information criterion AIC (Akaike, 1974) and Bayesian information criterion BIC (Schwarz, 1978) for model choice. For these data, AIC and BIC shows a better fit of the GM model. Standard deviations (SD) were computed using the inverse of the Hessian matrix. Figure 4 depicts the histogram for the data with the fitted densities, revealing good performance of the GM model.

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4.2 Energy consumption data

Devore (2005) present a set data associated to energy consumption (in BTU) of 90 homes with gas heating. The electric companies consider essential analyze this type information for respond to energy demands. Table (4) presents summary statistics for the energy consumption data where b_1 and b_2 are the coefficients of asymmetry and kurtosis, respectively.

Using results in Section 3.1, moment estimators were computed, leading to $\widehat{\theta} = 0.028$ and $\widehat{\alpha} = 2.300$. These estimates were then used as starting values for the optim algorithm for maximizing the likelihood function. Table 5 presents parameter estimates for the M, G, W and GM models, using maximum likelihood (MLE) approach and the corresponding Akaike information criterion AIC (Akaike, 1974) and Bayesian information criterion BIC (Schwarz, 1978) for model choice. For these data, AIC and BIC shows a better fit of the GM model. Standard deviations (SD) were computed using the inverse of the Hessian matrix. Figure 5 depicts the histogram for the data with the fitted densities, revealing good performance of the GM model.

5 Concluding remark

In this paper we study an extension of the Maxwell distribution more flexibility in terms of the asymmetry and kurtosis of distribution. This model is generated using the gamma-G generator of distributions. Moment and maximum likelihood estimators for the gamma-Maxwell distribution requires numerical procedures (such as the Newton-Raphson algorithm) to be computed. Applications to real data have demonstrated that the gamma-Maxwell distribution can present better fit than Maxwell distribution.

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Table 1: Maximum likelihood and Moment estimators for samples generated for several values of the parameters θ and α.

Maximum Likelihood Estimators							
Parameters		n = 50		n = 100		n = 200	
θ	α	θ (SD)	α (SD)	θ (SD)	α (SD)	θ (SD)	α (SD)
1.0	0.5	1.0930 (0.2499)	0.5325 (0.0977)	1.0398 (0.1666)	0.5160 (0.0671)	1.0167 (0.1192)	0.5065 (0.0467)
	1.5	1.0668 (0.2207)	1.6000 (0.3439)	1.0296 (0.1409)	1.5480 (0.2164)	1.0212 (0.1068)	1.5290 (0.1633)
	2.5	1.0680 (0.2259)	2.6846 (0.6113)	1.0306 (0.1521)	2.5917 (0.3974)	1.0230 (0.1015)	2.5535 (0.2744)
2.0	2.0	2.1430 (0.4621)	2.1534 (0.4861)	2.0374 (0.3015)	2.0439 (0.3136)	2.0265 (0.2038)	2.0288 (0.2120)
	3.0	2.1517 (0.4789)	3.2388 (0.7782)	2.0630 (0.3000)	3.1086 (0.4879)	2.0247 (0.2087)	3.0458 (0.3330)
	4.0	2.1353 (0.4446)	4.2827 (0.9540)	2.0838 (0.3072)	4.1817 (0.6683)	2.0383 (0.2170)	4.0828 (0.4617)
3.0	2.0	3.2043 (0.6496)	2.1504 (0.4703)	3.1015 (0.4301)	2.0734 (0.2990)	3.0467 (0.2989)	2.0301 (0.2095)
	3.0	3.2135 (0.6856)	3.2238 (0.7344)	3.0708 (0.4443)	3.0799 (0.4831)	3.0376 (0.3047)	3.0432 (0.3343)
	4.0	3.2122 (0.7050)	4.2947 (0.9950)	3.0952 (0.4625)	4.1325 (0.6525)	3.0413 (0.3069)	4.0611 (0.4383)
Moments Estimators							
1.0	0.5	1.0102 (0.2666)	0.4986 (0.1024)	0.9947 (0.1784)	0.4973 (0.0728)	0.9948 (0.1278)	0.4970 (0.0521)
	1.5	1.0703 (0.2287)	1.6066 (0.3605)	1.0305 (0.1469)	1.5496 (0.2262)	1.0236 (0.1099)	1.5330 (0.1688)
	2.5	1.1094 (0.6151)	2.6884 (1.6136)	1.2535 (3.9586)	2.7594 (0.9654)	1.114 (0.4051)	2.7371 (0.7549)
2.0	2.0	2.3666 (0.9839)	2.3517 (0.9526)	2.1557 (0.5953)	2.1607 (0.5830)	2.0781 (0.2812)	2.0851 (0.3025)
	3.0	1.8887 (0.6740)	2.7809 (0.9983)	1.9658 (0.6237)	2.8860 (0.7444)	2.0310 (0.5701)	3.0611 (0.6822)
	4.0	1.9370 (0.8780)	3.7745 (2.0812)	2.0621 (0.8622)	3.8707 (0.8108)	1.9540 (0.6196)	3.9063 (0.8024)
3.0	2.0	3.4461 (1.1054)	2.3961 (1.0311)	3.2651 (0.7879)	2.2113 (0.6735)	3.1141 (0.3993)	2.0790 (0.2895)
	3.0	2.8718 (1.0959)	2.8570 (1.1286)	3.1219 (1.0516)	3.0388 (1.0692)	3.1327 (0.8678)	3.1439 (0.8875)
	4.0	2.8133 (0.7490)	3.7607 (1.1259)	2.8594 (0.7067)	3.7988 (0.9197)	2.8846 (0.6917)	3.8438 (0.8850)

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Table 2: Summary statistics for wind speed data set.

n	$\overline{X}$	s^2	b_1	b_2
41323	2.355	2.231	0.705	3.259

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Table 3: Estimated Parameters of the M, G, W and GM distributions for wind speed data set.

Model	Parameter estimates (SD)	AIC	BIC
M	$\widehat{\theta} = 0.448$ (0.003)	163700.9	163709.6
G	$\widehat{\theta} = 0.851$ (0.006)	143795.6	143812.9
	$\widehat{\alpha} = 2.005$ (0.012)		
W	$\widehat{\theta} = 2.619$ (0.008)	142394.7	142411.9
	$\widehat{\alpha} = 1.582$ (0.006)		
GM	$\widehat{\theta} = 0.096$ (0.001)	141992.1	142009.4
	$\widehat{\alpha} = 0.448$ (0.003)		

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Table 4: Summary statistics for energy consumption data set.

n	$\overline{X}$	s^2	b_1	b_2
90	10.038	8.225	0.283	3.000

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Table 5: Estimated Parameters of the M and GM distributions for energy consumption data set.

Model	Parameter estimates (SD)	AIC	BIC
M	$\widehat{\theta} = 0.013$ (0.001)	467.900	470.400
G	$\widehat{\theta} = 1.153$ (0.173)	448.837	453.836
	$\widehat{\alpha} = 11.579$ (1.701)		
W	$\widehat{\theta} = 11.093$ (0.325)	449.443	454.442
	$\widehat{\alpha} = 3.792$ (0.300)		
GM	$\widehat{\theta} = 0.027$ (0.004)	447.158	452.157
	$\widehat{\alpha} = 2.271$ (0.355)		

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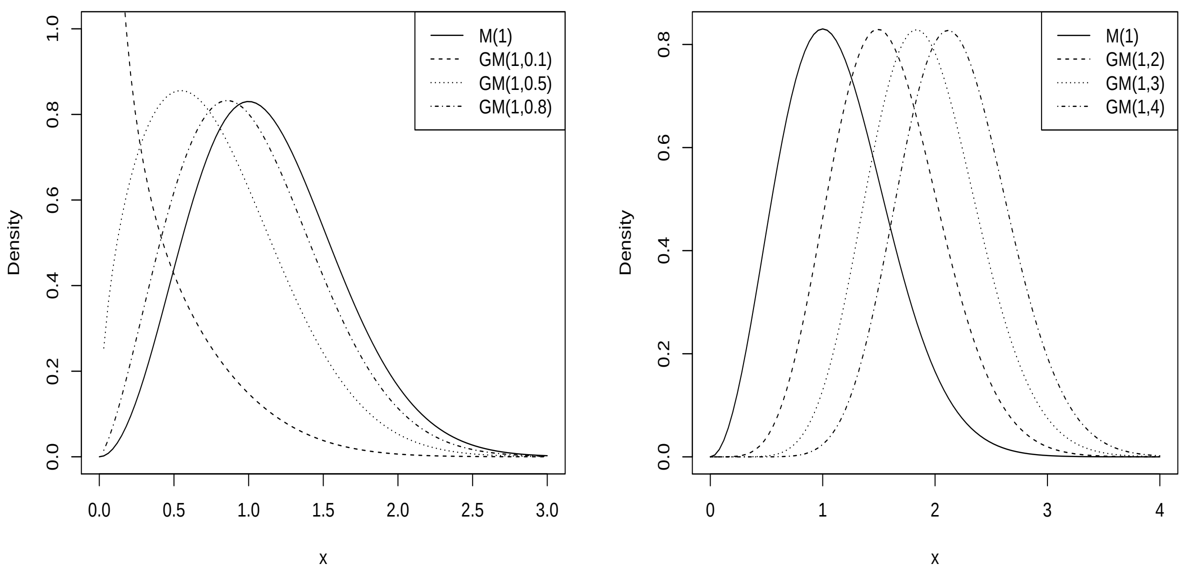

Figure 1: Plot of the Gamma-Maxwell density, $GM(\theta, \alpha)$.

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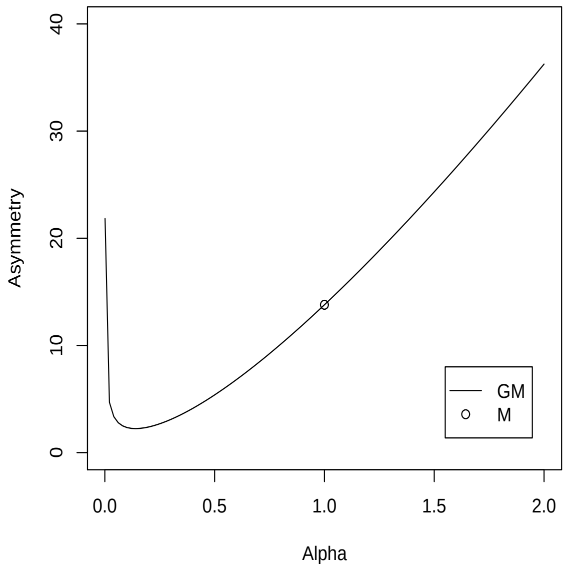

Figure 2: Asymmetry coefficient.

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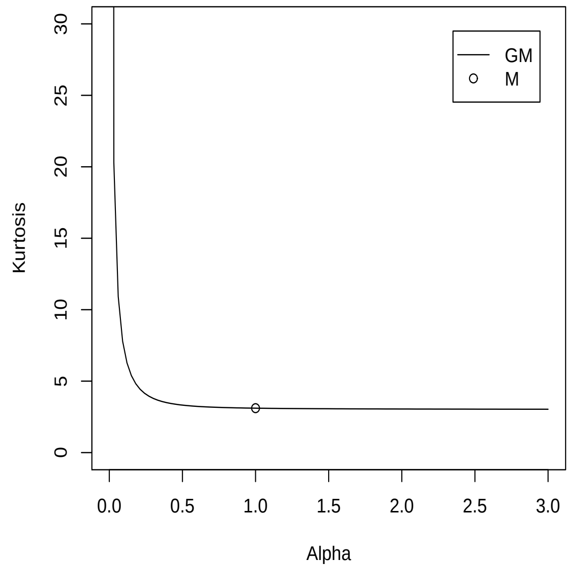

Figure 3: Kurtosis coefficient

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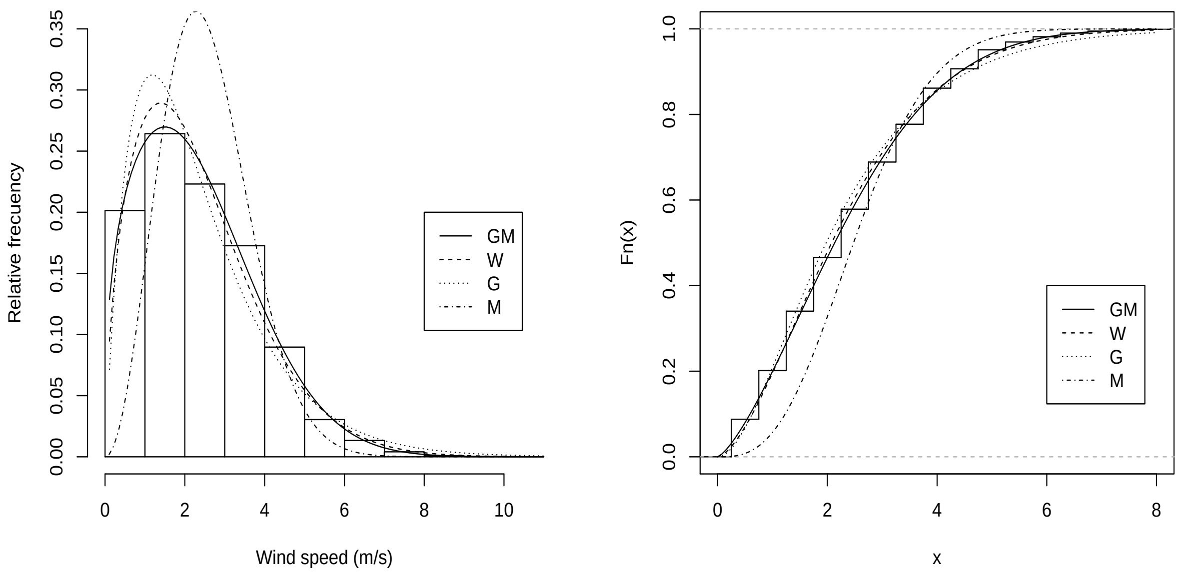

Figure 4: Models fitted by maximum likelihood method for wind speed data set.

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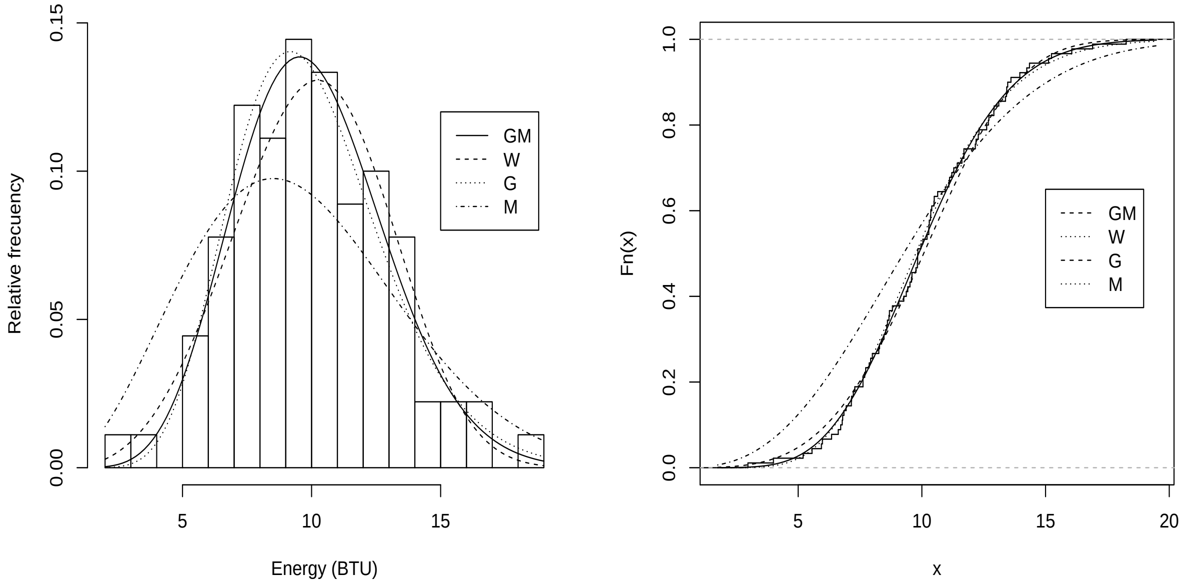

Figure 5: Models fitted by maximum likelihood method for energy consumption data set.

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