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On the convergence of spectral approximations for the heat convection equations

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Abstract In this paper, we focus on the convergence rate of solutions of spectral Galerkin approximations for the heat convection equations on a bounded domain. Estimates in H^2-norm for velocity and temperature without compatibility conditions are obtained. Moreover, we give rates of convergence for the velocity and temperature derivatives in L^2-norm.

Keywords Spectral Galerkin approximations · Convergence rate · Boussinesq equations · Navier-Stokes type equations · Heat convection equations

Mathematics Subject Classification 65M60 · 76D05 · 65M15 · 76R99

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1 Introduction

The study of the existence of solutions to the Navier-Stokes equations can be made demanding only that the basis functions generate a complete subspace in a suitable space of functions, see Kiselev and Ladyzehnskaya [17], Heywood [9] or the classical books of Ladyzhenskaya [19], Lions [20] or Temam [41]. To obtain refinements in the theory, particularly those concerning the regularity and decay, an appropriate choice of the base generally is required. For example, the eigenfunctions of Stokes operator associated with the problem could be taken.

An interesting question from the point of view of theoretical and numerical analysis is to determine the convergence rates in several norms of the difference between the exact solution and Galerkin approximations. This was done by Rautmann [26]. These error estimates are local in the sense that they depend on functions that grow exponentially with time. Rautmann obtained the optimal convergence rate in the H^1-norm, but he only improved L^2-error estimates when compared with the trivial one that derives directly from the H^1-estimate. The optimal convergence rate in the L^2-norm was obtained by Salvi [38] (see also [31]). We pointed that higher-order error estimates is a difficult question because, as it was observed by Heywood and Rannacher [10], they depend on non-local compatibility conditions for the data at time $t = 0$, which cannot be verified in practice. In this direction Rautmann [27] studied how smooth a Navier-Stokes solution can be at time $t = 0$ without any compatibility condition mentioned above (see also Temam [42] for other formulation of the compatibility condition). By using this, Rautmann [28] proves an error estimate in the H^2-norm. This is the best estimate that we may expect without any assumptions about the stability of the solution being approximated (see [8]). For the classical Navier-Stokes equations, assuming uniform boundedness in time of the L^2-norm of the gradient of the velocity and exponential stability in the Dirichlet norm of the solution, optimal uniform-in-time error estimates for the velocity in the Dirichlet norm were derived in [8]. An optimal uniform-in-time error estimate for the velocity in the L^2 norm was derived in [38], also for the classical Navier-Stokes equations, assuming exponential stability in the L^2 norm. Rojas-Medar and Boldrini [31] proved uniform in time optimal error estimates for the spectral Galerkin approximations in the H^1 and L^2 norms, considering the external force field has a mild form of decay, without explicitly assuming the L^2 (or H^1)-exponential stability (this being in general difficult to verify).

An extension to the more practical finite-element approximations was analyzed intensively by Heywood and Rannacher in a series of beautiful papers [10–13], see also the paper of Bause [2] and the references therein.

We point that the exact knowledge of the eigenfunctions of the Stokes operator is possible in certain domains, see [36,37]. Moreover, the asymptotic behaviour of the eigenvalues is well known, see [16] and the references therein.

In this work, we want to study the convergence rates for the spectral Galerkin approximations for the heat convection equations, also called Boussinesq equations or Oberbeck-Boussinesq. We recall the results given in the works [32–34], where the existence, regularity and uniqueness of solutions are established by means of the spectral Galerkin method and the estimates for Galerkin approximations necessary for our future arguments, see also [18,23,24]. Other techniques utilized in the study of these equations are semigroups [7,30], hydrodynamic potentials [3,39], iterative methods [22], for instance.

Analogous convergence rates to those given by Rautmann [26] and Boldrini and Rojas-Medar [31] were derived in the work [35] for a system more general than that of Boussinesq's. Analogous results to those of Heywood [8] were given in [4] for the Boussinesq model. We recall these results in Sect. 2 since they will be used in this paper.

We observe that the main results given in [43] are not new (Theorem 3.1 in [43] was established in [35]). The Theorem 3.2 in [43] is not optimal, they set strong conditions on the external forces and assume null initial conditions (see Rautmann [28]).

In this work, we extend the results by Rautmann [27,28] to the Boussinesq system; we prove the pointwise convergence rate in the H^2-norm for the velocity and temperature. Moreover, the pointwise convergence rate in the L^2-norm for the time-derivative of velocity and temperature is obtained. The innovation of our results is again, that we do not need impose compatibility conditions on the initial data.

As it is usual in this context, to simplify the notation, we will denote by C generic finite positive constant depending only on Ω and the other fixed parameters of the problem (like the initial data) that may have different values in different expressions.

2 Preliminaries

The following equations describe the heat convection motion of a fluid in a bounded domain $\Omega \subset \mathbb{R}^N$, $N = 2$ or 3, with smooth boundary, in the time interval $[0, T)$, $0 \leq T \leq \infty$, considering the Oberbeck-Boussinesq approximation (see Joseph [15]):

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{j} + \theta \boldsymbol{g}, \\ \operatorname{div} \boldsymbol{u} = 0, \\ \dfrac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla \theta - k \Delta \theta = f. \end{cases} \tag{1}$$

Here $\boldsymbol{u}(t, x) \in \mathbb{R}^N$, $\theta(t, x) \in \mathbb{R}$, and $p(t, x) \in \mathbb{R}$ denote respectively the unknowns velocity, temperature and pressure of a liquid at a point $x \in \Omega$ at time $t \in [0, T)$. The constants ν and k are respectively, the kinematic viscosity and thermal conductivity. The gravitational field, $\boldsymbol{g}(t, x)$, the coefficient of volume expansion, $\boldsymbol{j}(t, x)$, and the source function $f(t, x)$ are given.We have considered the coefficient of viscosity and thermal conductivity equal to 1, without loss of generality.

On the boundary Γ, we assume that

$$\boldsymbol{u}(t, x) = 0, \;\; \theta(t, x) = \theta_1, \tag{2}$$

where θ_1 is a known function, and the initial data conditions are expressed by

$$\boldsymbol{u}(0, x) = \boldsymbol{u}_0(x), \;\; \theta(0, x) = \theta_0(x), \tag{3}$$

where $\boldsymbol{u}_0$ and θ_0 are given functions on the variable $x \in \Omega$.

To simplify the analysis, we consider $\nu = 1$, $k = 1$, and $\theta_1 = 0$. The nonhomogeneous case $\theta_1 \neq 0$, can be treated by using an appropriate lifting and only the obvious changes should be required in the statement of the results.

The expressions ∇, Δ and div denote the gradient, Laplacian and divergence operators, respectively (we also denote $\dfrac{\partial \boldsymbol{u}}{\partial t}$ by $\boldsymbol{u}_t$); the ith component of $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ is given by $[(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}]_i = \sum_j u_j \frac{\partial u_i}{\partial x_j}$ and $\boldsymbol{u} \cdot \nabla\theta = \sum_j u_j \frac{\partial \theta}{\partial x_j}$.

We will consider the usual Sobolev spaces

$$W^{m,q}(D) = \{f \in L^q(D),\ ||\partial^\alpha f||_{L^q(D)} < +\infty,\ |\alpha| \leq m\},$$

$m = 0, 1, 2, \ldots, 1 \leq q \leq +\infty$, $D = \Omega$ or $\Omega\times]0, T[$, $0 < T < +\infty$, with the usual norm. When $q = 2$, we denote by $H^m(D) = W^{m,2}(D)$ and $H_0^m(D) =$ closure of $C_0^\infty(D)$ in $H^m(D)$. The L^q-norm is denoted by $\|\cdot\|_p$. When $q = 2$, the L^2-norm is denoted by $\|\cdot\|$ and the associated inner product in $L^2(\Omega)$ by $(\cdot, \cdot)$. If X is a Banach space, we denote by $L^q(0, T; X)$ the Banach space of the X-valued functions defined in the interval $[0, T]$ that are L^q-integrable in the sense of Bochner. In addition, boldface letters will be used for vectorial spaces.

We shall consider the following spaces of divergence free functions

$$\begin{aligned} \boldsymbol{C}_{0,\sigma}^\infty(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{C}_0^\infty(\Omega) |\ \text{div}\ \boldsymbol{v} = 0 \text{ in } \Omega\}, \\ \boldsymbol{H} &= \text{ closure of } \boldsymbol{C}_{0,\sigma}^\infty(\Omega) \text{ in } \boldsymbol{L}^2(\Omega), \\ \boldsymbol{V} &= \text{ closure of } \boldsymbol{C}_{0,\sigma}^\infty(\Omega) \text{ in } \boldsymbol{H}^1(\Omega). \end{aligned}$$

Throughout the paper, P denotes the orthogonal projection from $\boldsymbol{L}^2(\Omega)$ into $\boldsymbol{H}$ and $A = -P\Delta$ with $D(A) = \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$ is the usual Stokes operator.

We will denote by $\boldsymbol{w}^n(x)$ and λ_n the eigenfunctions and eigenvalue of A. It is well know that $\{\boldsymbol{w}^n\}_{n=1}^\infty$ form an orthogonal complete system in the spaces $\boldsymbol{H}$, $\boldsymbol{V}$ and $\boldsymbol{H}^2(\Omega) \cap \boldsymbol{V}$, with their usual inner products $(\boldsymbol{u}, \boldsymbol{v})$, $(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})$ and $(A\boldsymbol{u}, A\boldsymbol{v})$ respectively.

We observe that for the regularity of the Stokes operator, it is usually assumed that Ω is of class C^3; this being in order to use Cattabriga's results [5]. However, we use the stronger results of Amrouche and Girault [1], which implies, in particular, that when $A\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$, then $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ and $\|\boldsymbol{u}\|_{H^2}$ and $\|A\boldsymbol{u}\|$ are equivalent norms when Ω is of class $C^{1,1}$.

For each $n \in \mathbb{N}$, we denote by P_n the orthogonal projections from $\boldsymbol{L}^2(\Omega)$ onto $\boldsymbol{V}_n = \text{span}\{\boldsymbol{w}^1(x), \ldots, \boldsymbol{w}^n(x)\}$. For more details on the Stokes operator see Temam [41].

Similar considerations are true for the Laplacian operator $B = -\Delta : D(B) \to L^2(\Omega)$ with the Dirichlet boundary conditions with domain $D(B) = H_0^1(\Omega) \cap H^2(\Omega)$ and we will denote by $\omega^n(x)$, γ_n the eigenfunctions and eigenvalues of B. Also we denote $H_n = \text{span}\ \{\omega^1(x), \ldots, \omega^n(x)\}$ and R_n the orthogonal projections from $\boldsymbol{L}^2(\Omega)$ onto H_n. We can rewrite the problem (1) by using the orthogonal projection P as follows

$$\begin{cases} \boldsymbol{u}_t + A\boldsymbol{u} + P(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) = P(\boldsymbol{j} + \theta\boldsymbol{g}), \\ \theta_t + B\theta + \boldsymbol{u}\cdot\nabla\theta = f, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \theta(0) = \theta_0, \end{cases} \tag{4}$$

which is equivalent to the weak form

$$\begin{cases} (\boldsymbol{u}_t, \boldsymbol{v}) + (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v}) + (\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) = (\boldsymbol{j}, \boldsymbol{v}) + (\theta\boldsymbol{g}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \\ (\theta_t, \rho) + (\boldsymbol{u}\cdot\nabla\theta, \rho) + (\nabla\theta, \nabla\rho) = (f, \rho) \quad \forall \rho \in H_0^1(\Omega), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0,\ \theta(0) = \theta_0. \end{cases} \tag{5}$$

The spectral Galerkin approximations for $(\boldsymbol{u}, \theta)$ are defined for each $n \in \mathbf{N}$ as the solution $(\boldsymbol{u}^n, \theta^n) \in C^2([0,T]; \boldsymbol{V}_n) \times C^2([0,T]; H_n)$ of

$$\begin{cases} (\boldsymbol{u}^n_t, \boldsymbol{v}) + (\boldsymbol{u}^n\cdot\nabla\boldsymbol{u}^n, \boldsymbol{v}) + (\nabla\boldsymbol{u}^n, \nabla\boldsymbol{v}) = (\boldsymbol{j}, \boldsymbol{v}) + (\theta^n\boldsymbol{g}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in V_n, \\ (\theta^n_t, \rho) + (\nabla\theta^n, \nabla\rho) + (\boldsymbol{u}^n\cdot\nabla\theta^n, \rho) = (f, \rho) \qquad \forall \rho \in H_n, \\ \boldsymbol{u}^n(x,0) = P_n\boldsymbol{u}_0(x),\ \theta^n(x,0) = R_n\theta_0(x), \quad x \in \Omega. \end{cases} \tag{6}$$

By using these approximations Rojas-Medar and Lorca [32] have proved the following result:

Theorem 1 *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary Γ of class $C^{1,1}$. Suppose that*

$$(\boldsymbol{u}_0, \theta_0) \in \boldsymbol{V} \times H_0^1(\Omega), \tag{7}$$

$$\boldsymbol{j} \in L^2(0,T; \boldsymbol{L}^2(\Omega)), \quad \boldsymbol{g} \in L^2(0,T; \boldsymbol{L}^3(\Omega)), \quad f \in L^2(0,T; L^2(\Omega)). \tag{8}$$

Then, there exists $T_0 > 0$ with $T_0 \le T$ such that the problem (1) has a unique solution in the interval $[0, T_0)$. This solution belongs to $C([0,T_0); \boldsymbol{V}) \times C([0,T_0); H_0^1(\Omega))$. Moreover, the approximations of spectral Galerkin satisfy the following estimates uniform in n

$$\|\nabla\boldsymbol{u}^n(t)\|^2 + \|\nabla\theta^n(t)\|^2 \le C, \tag{9}$$

$$\int_0^t (\|A\boldsymbol{u}^n(\tau)\|^2 + \|B\theta^n(\tau)\|^2)\, d\tau \le C, \tag{10}$$

$$\int_0^t (\|\boldsymbol{u}^n_t(\tau)\|^2 + \|\theta^n_t(\tau)\|^2)\, d\tau \le C. \tag{11}$$

With stronger assumptions on the initial values and the external fields, we are able to prove the following theorem:

Theorem 2 *Under the hypotheses of Theorem 1, if moreover the forces satisfy*

$$\mathfrak{F}_1(t) = \int_0^T (\|\boldsymbol{g}_t(s)\|^2 + \|\boldsymbol{j}_t(s)\|^2 + \|f_t(s)\|^2) ds < +\infty$$

and the initial data $\boldsymbol{u}_0 \in D(A)$, $\theta_0 \in D(B)$ then, the solution $(\boldsymbol{u}, \theta)$ obtained in Theorem 1 belongs to $C([0,T_0]; D(A)\times D(B)) \cap C^1([0,T_0]; \boldsymbol{H} \times (L^2(\Omega)))$. Furthermore, the approximations $\boldsymbol{u}^n$, θ^n satisfy

$$\|\boldsymbol{u}_t^n(t)\|^2 + \|\theta_t^n(t)\|^2 \le C, \tag{12}$$

$$\|A\boldsymbol{u}^n(t)\|^2 + \|B\theta^n(t)\|^2 \le C, \tag{13}$$

$$\int_0^t \|\nabla \boldsymbol{u}_t^n(\tau)\|^2 + \|\nabla\theta_t^n(\tau)\|^2 d\tau \le C. \tag{14}$$

Remark 1 Moreover, it was proved in [34] the global existence and uniqueness of strong solutions. We note that it achieves the same results as in the case of classical Navier-Stokes equations, i.e., without smallness of the forces and initial data for $n = 2$, and smallness if $n = 3$.

In the work [35] the following optimal results were proved for the rate of convergence in the L^2 and H^1-norms.

Theorem 3 *Suppose the assumptions of the Theorem 2 hold. Then, the approximation* $\boldsymbol{u}^n$ *and* θ^n *satisfies*

$$\begin{aligned} &\|\boldsymbol{u}(t) - \boldsymbol{u}^n(t)\|^2 + \|\theta(t) - \theta^n(t)\|^2 \\ &\quad + \int_0^t (\|\nabla \boldsymbol{u}(s) - \nabla \boldsymbol{u}^n(s)\|^2 + \|\nabla\theta(s) - \nabla\theta^n(s)\|^2)\, ds \\ &\quad \le C\left(\frac{1}{\lambda_{n+1}^2} + \frac{1}{\gamma_{n+1}^2}\right). \end{aligned} \tag{15}$$

$$\begin{aligned} &\|\nabla \boldsymbol{u}(t) - \nabla \boldsymbol{u}^n(t)\|^2 + \|\nabla\theta(t) - \nabla\theta^n(t)\|^2 \\ &\quad + \int_0^t (\|\boldsymbol{u}_t(s) - \boldsymbol{u}_t^n(s)\|^2 + \|\theta_t(s) - \theta_t^n(s)\|^2)\, ds \le C\left(\frac{1}{\lambda_{n+1}} + \frac{1}{\gamma_{n+1}}\right). \end{aligned} \tag{16}$$

$$\|\boldsymbol{u}_t(t) - \boldsymbol{u}_t^n(t)\|_{\boldsymbol{V}^*}^2 + \|\theta_t(t) - \theta_t^n(t)\|_{H^{-1}}^2 \le C\left(\frac{1}{\lambda_{n+1}} + \frac{1}{\gamma_{n+1}}\right). \tag{17}$$

$$\|A\boldsymbol{u}(t) - A\boldsymbol{u}^n(t)\|_{\boldsymbol{V}^*}^2 + \|B\theta(t) - B\theta^n(t)\|_{H^{-1}}^2 \le C\left(\frac{1}{\lambda_{n+1}} + \frac{1}{\gamma_{n+1}}\right). \tag{18}$$

3 The main result

From now on, for simplicity of notation, we will write $T_0 \equiv T$.

Theorem 4 *Under the assumptions of Theorem 2, if moreover* $\boldsymbol{g}, \boldsymbol{j} \in C([0,T], \boldsymbol{H}^1(\Omega))$, $f \in C([0,T], H^1(\Omega))$ *and* $\boldsymbol{u}_0 \in D(A^{1+\varepsilon})$, $\theta_0 \in D(B^{1+\varepsilon})$ *with* $\varepsilon \in (0, \frac{1}{4})$, *then,*

$$\|A\boldsymbol{u}(t) - A\boldsymbol{u}^n(t)\| + \|\boldsymbol{u}_t(t) - \boldsymbol{u}_t^n(t)\| \le C\left[\frac{C(\varepsilon)}{\lambda_{n+1}^\varepsilon} + \left(\frac{1}{\lambda_{n+1}} + \frac{1}{\gamma_{n+1}}\right)^{1/2}\right], \tag{19}$$

$$\|B\theta(t) - B\theta^n(t)\| + \|\theta_t(t) - \theta_t^n(t)\| \le C\left[\frac{C(\varepsilon)}{\lambda_{n+1}^\varepsilon} + \frac{1}{\gamma_{n+1}^\varepsilon} + \left(\frac{1}{\gamma_{n+1}} + \frac{1}{\gamma_{n+1}}\right)^{1/2}\right]. \tag{20}$$

4 Technical results

In the following, we will use fractional powers A^α and B^α defined for any real α by means of the spectral representation of A and B respectively, (see [40] for the Stokes operator). On $D(A^\alpha)$, the operator A^α commute with $\exp(-tA)$ and analogously for B^α. The following properties, valid for any strictly positive, self-adjoint operator $\mathcal{A}$ in Hilbert space $\mathcal{H}$, will be used (see [6])

$$\|\mathcal{A}^\alpha \exp(-t\mathcal{A})\| \le t^{-\alpha} \quad \text{with } t > 0,\ 0 \le \alpha \le \text{e}. \tag{21}$$

$$\|\mathcal{A}^{\alpha+\beta} \exp(-t\mathcal{A})v\| = \|\mathcal{A}^\alpha \exp(-t\mathcal{A})\mathcal{A}^\beta v\| \le t^{-\alpha}\|\mathcal{A}^\beta v\| \tag{22}$$

for $v \in D(\mathcal{A}^\beta)$, $t > 0$ and $0 \le \alpha \le$ e.

$$\|(\exp(-t\mathcal{A}) - I)v\| \le \frac{t^\sigma}{\sigma}\|\mathcal{A}^\sigma v\| \quad \text{with } v \in D(\mathcal{A}^\sigma),\ t > 0,\ 0 < \sigma < 1. \tag{23}$$

$$\|(\exp(-t\mathcal{A}) - I)v\| \longrightarrow 0^+ \quad \text{when } t \to 0, \quad v \in \mathcal{H}. \tag{24}$$

The following lemmas will be used below.

Lemma 1 *Assume $\boldsymbol{u}_i \in D(A^{3/4+\eta})$ for some $\eta > 0$ and $\boldsymbol{v}_i \in D(A)$ for $i = 1, 2$. Then,*

$$\begin{aligned}\|A^\zeta P(\boldsymbol{u}_2 \cdot \nabla \boldsymbol{v}_2 - \boldsymbol{u}_1 \cdot \nabla \boldsymbol{v}_1)\| \le\ & C\|A^{3/4+\eta}(\boldsymbol{u}_2 - \boldsymbol{u}_1)\|\|A\boldsymbol{v}_2\| \\ & + C\|A^{3/4+\eta}\boldsymbol{u}_1\|\|A(\boldsymbol{v}_2 - \boldsymbol{v}_1)\|\end{aligned}$$

holds for all $\zeta \in [0, 1/4)$, the constant C depending only on η and ζ.

This result can be found in [29], Corollary 3.4. In analogous way, the following lemma can be proved.

Lemma 2 *Assume $\boldsymbol{u}_i \in D(A^{3/4+\eta})$ for some $\eta > 0$ and $\theta_i \in D(B)$ for $i = 1, 2$. Then,*

$$\begin{aligned}\|B^\zeta R(\boldsymbol{u}_2 \cdot \nabla \theta_2 - \boldsymbol{u}_1 \cdot \nabla \theta_1)\| \le\ & C\|A^{3/4+\eta}(\boldsymbol{u}_2 - \boldsymbol{u}_1)\|\|B\theta_2\| \\ & + C\|A^{3/4+\eta}\boldsymbol{u}_1\|\|B(\theta_2 - \theta_1)\|\end{aligned}$$

holds for all $\zeta \in [0, 1/4)$, the constant C depending only on η and ζ.

Lemma 3 *Let T, C_1, and C_2 be positive constants and let r be a constant with $0 < r < 1$. Then, any continuous positive function f, defined for $t \in [0, T]$, satisfying*

$$f(t) \le C_1 + C_2 \int_0^t (t-s)^{-r} f(s)ds,$$

verifies

$$f(t) \le CC_1 \exp\left(CC_2^{1/(1-r)}t\right)$$

with a positive constant C, which depends only on r.

This result can be found in [25], Lemma 6.5, and the following lemma in [14], p. 38.

Lemma 4 *In a Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ defining the norm $|\cdot|_H$, let $A^\star$ be a symmetric operator which has the complete orthonormal system of eigenfunction (e_i^*) corresponding to the sequence (λ_i^*) of eigenvalues $0<\lambda_1^*\leq\lambda_1^*\leq\cdots\leq\lambda_i^*\to\infty$ with $i\to\infty$. then the error estimate*

$$|f-\sum_{i=1}^{k}\langle f,e_i^*\rangle e_i^*|_H\leq(\lambda_{k+1}^*)^{-1}|A^\star f|_H$$

holds for any $f\in D(A^\star)$.

We going to prove some results of regularity for the solution obtained in the Theorems 1 and 2. Firstly, observe that we can write the following representation of the solution obtained in Theorem 1

$$\begin{aligned}\boldsymbol{u}(t)&=\exp(-At)\boldsymbol{u}_0\\&\quad+\int_0^t\exp(-(t-s)A)(P(\boldsymbol{j}(s)+\theta(s)\boldsymbol{g}(s))-P(\boldsymbol{u}(s)\cdot\nabla\boldsymbol{u}(s)))ds,\\\theta(t)&=\exp(-Bt)\theta_0+\int_0^t\exp(-(t-s)B)(f(s)-\boldsymbol{u}(s)\cdot\nabla\theta(s))ds.\end{aligned}$$

Theorem 5 *Suppose that $\boldsymbol{g},\boldsymbol{j}\in C([0,T],\boldsymbol{H}^1(\Omega))$, $f\in C([0,T],H^1(\Omega))$ and $\boldsymbol{u}_0\in D(A^{1+\varepsilon})$, $\theta_0\in D(B^{1+\varepsilon})$, then, the solution $(\boldsymbol{u},\theta)$ of (1) satisfies*

$$\boldsymbol{u}\in C([0,T];D(A^{1+\varepsilon}))\cap C^1([0,T];D(A^\varepsilon)),\tag{25}$$

$$\theta\in C([0,T];D(B^{1+\varepsilon})\cap C^1([0,T];D(B^\varepsilon)),\tag{26}$$

for $0\leq\varepsilon<1/4$.

Proof The proof is similar to that of Bause [2] (Theorem 3.11). In fact, it is exactly equal in the case of the velocity $\boldsymbol{u}(t)$. To the temperature θ, we sketch the main ideas following [2]

Step 1: If $0\leq\varepsilon<1/4$ then $D(B^{1+\varepsilon})=H^{2+2\varepsilon}(\Omega)\cap H_0^1(\Omega)$.

Let $v\in H^{2+2\varepsilon}(\Omega)\cap H_0^1(\Omega)$, then $Bv\in H^{2\varepsilon}(\Omega)$. Since $H^{2\varepsilon}(\Omega)=D(B^\varepsilon)$ (see [2, lemma 3.4]) it follows that $v\in D(B^{1+\varepsilon})$. Moreover the injection is compact because $\|B^{1+\varepsilon}v\|\leq C\|Bv\|_{2\varepsilon}\leq C\|v\|_{2+2\varepsilon}$. Conversely, if $v\in D(B^{1+\varepsilon})$, then $v\in H^{2+2\varepsilon}(\Omega)$ and, using the fact that for the Dirichlet problem (with $\partial\Omega\in C^{k+2}$, $k=0,1$)

$$-\triangle\psi=g\text{ in }\Omega,\quad\psi|_{\partial\Omega}=0,$$

the solution satisfies $\|\psi\|_{H^{k+2}}\leq C\|g\|_{H^k}$, $k=0,1$, (see V. Mikhailov [21], for instance), we have that $\|v\|_{2+2\varepsilon}\leq C\|Bv\|_{2\varepsilon}\leq C\|B^{1+\varepsilon}v\|$.

Step 2: If $0\leq\varepsilon<1/4$ then $(\boldsymbol{u}\cdot\nabla)\theta\in C([0,T];D(B^\varepsilon))$

Indeed, $\boldsymbol{u} \in D(A)$ and $\theta \in D(B)$ imply that $(\boldsymbol{u} \cdot \nabla)\theta \in H^1(\Omega)$ and

$$\begin{aligned}\|(\boldsymbol{u}\cdot\nabla)\theta\|_{H^1} &\le \|(\boldsymbol{u}\cdot\nabla)\theta\| + \|(\nabla\boldsymbol{u}\cdot\nabla)\theta\| + \|(\boldsymbol{u}\cdot\nabla(\nabla\theta)\| \\ &\le C(\|\boldsymbol{u}\|_\infty\|\nabla\theta\| + \|\nabla\boldsymbol{u}\|_4\|\nabla\theta\|_4 + \|\boldsymbol{u}\|_\infty\|\theta\|_{H^2}) \\ &\le C\|A\boldsymbol{u}\|\|B\theta\|.\end{aligned}$$

Here, we have used the inequalities

$$\begin{aligned}&\|v\|_4 \le C\|\nabla(\nabla v)\|^{3/4}\|\nabla v\|^{1/4}, \quad \|v\|_\infty \le C\|v\|_{H^2}, \\ &\|v\|_{H^2} \le C\|Av\| \text{ with } v \in D(A), \quad \|v\|_{H^2} \le C\|Bv\| \text{ for } v \in D(B).\end{aligned}$$

Moreover, since $H^1(\Omega) \hookrightarrow H^{2\varepsilon}(\Omega)$ is continuous, we get that $(\boldsymbol{u} \cdot \nabla)\theta \in D(B^\varepsilon)$, $0 \le \varepsilon < 1/4$.

To the continuity of term $(\boldsymbol{u} \cdot \nabla\theta)(t)$, observe that

$$\begin{aligned}\|B^\varepsilon(\boldsymbol{u}\cdot\nabla\theta)(t) - B^\varepsilon(\boldsymbol{u}\cdot\nabla\theta)(s)\| &\le C(\|(\boldsymbol{u}(t)-\boldsymbol{u}(s))\cdot\nabla\theta(t)\|_{H^1} \\ &\quad+\|\boldsymbol{u}(s)\cdot\nabla(\theta(t)-\theta(s))\|_{H^1}) \\ &\le C((\|B\theta(t)\| + \|A\boldsymbol{u}(s)\|)\|A(\boldsymbol{u}(t)-\boldsymbol{u}(s))\| \\ &\quad+\|A\boldsymbol{u}(s)\|\|B(\theta(t)-\theta(s))\|).\end{aligned}$$

Step 3: If $0 \le \varepsilon < 1/4$ then $\theta \in C([0, T]; D(B^{1+\varepsilon}))$.

By applying the operator $B^{1+\varepsilon}$ at both side of the integral equation of $\theta(t)$, (4), in the way

$$B^{1+\varepsilon}\theta(t) = \exp(-Bt)B^{1+\varepsilon}\theta_0 + \int_0^t B^\beta\exp(-(t-s)B)B^\sigma(f(s) - \boldsymbol{u}(s)\cdot\nabla\theta(s))ds,$$

where $\beta \in (0, 1)$, $\sigma \in (0, 1/4)$ such that $\beta + \sigma = 1 + \varepsilon$, using that $\|B^{1+\varepsilon}\theta_0\| \le \|\theta_0\|_{2+2\varepsilon}$,

$$\begin{aligned}&\left\|B^\beta\exp(-(t-s)B)B^\sigma(f(s) - \boldsymbol{u}(s)\cdot\nabla\theta(s))\right\| \\ &\quad\le \frac{1}{(t-s)^\beta}\|B^\sigma(f(s) - \boldsymbol{u}(s)\cdot\nabla\theta(s))\|\end{aligned}$$

and (21), we deduce that $\theta(t) \in D(B^{1+\varepsilon})$ for $t \in [0, T]$.

To prove the continuity, similar analysis can be apply to $B^{1+\varepsilon}\theta(t+h) - B^{1+\varepsilon}\theta(t)$ with $h \in \mathbb{R}$, getting, for example for $h > 0$, the following estimate:

$$\begin{aligned}&\left\|B^{1+\varepsilon}\theta(t+h) - B^{1+\varepsilon}\theta(t)\| \le \|(\exp(-hB) - I)B^{1+\varepsilon}\theta_0\right\| \\ &\quad+\int_0^t (t-s)^{-\beta}h^\delta(\|f(s)\|_{2(\delta+\sigma)} + \|(u\cdot\nabla\theta)(s)\|_{2(\delta+\sigma)})ds \\ &\quad+\int_0^{t+h}(t+h-s)^{-\beta}(\|f(s)\|_{2\sigma} + \|(u\cdot\nabla\theta)(s)\|_{2\sigma})ds\end{aligned}$$

$$\leq \|(\exp(-hB) - I)B^{1+\varepsilon}\theta_0\| + Ct^{1-\beta}h^{\delta} + Ch^{1-\beta}.$$

where $\delta > 0$ is such that $\delta + \sigma < 1/4$ and σ as above. Here, (22) and (23) are been used. The case $h < 0$ can be deal with in a similar way. Take in to account (24), it follows that $\theta \in C([0, T]; D(B^{1+\varepsilon}))$.
Step 4: By using the equation $\theta_t + B\theta + \boldsymbol{u} \cdot \nabla\theta = f$, of (4), we obtain that $\partial_t\theta \in C([0, T]; D(B^{\varepsilon}))$. □

5 H^2-error estimates for the velocity and the temperature

Let $\boldsymbol{u} = \sum_{i=1}^{\infty} A_i(t)\boldsymbol{w}^i(x)$ and $\theta = \sum_{i=1}^{\infty} B_i(t)\omega^i(x)$ the eigenfunctions expansion of $\boldsymbol{u}$, and θ, respectively. Let $\boldsymbol{v}^n = P_n\boldsymbol{u}$ and $\rho^n = R_n\theta$ the n-th partial sums of the series for $\boldsymbol{u}$ and θ, respectively. Recall that $(\boldsymbol{u}^n, \theta^n)$, solution of (6), are the spectral approximations for $(\boldsymbol{u}, \theta)$. We define

$$\mathrm{e}^n = \boldsymbol{u} - \boldsymbol{v}^n, \quad \varepsilon^n = \theta - \rho^n, \quad \boldsymbol{w}^n = \boldsymbol{v}^n - \boldsymbol{u}^n, \quad \eta^n = \rho^n - \theta^n.$$

To estimate $A\boldsymbol{u} - A\boldsymbol{u}^n$ and $B\theta - B\theta^n$, we have to estimate $A\mathrm{e}^n$, $B\varepsilon^n$, $A\boldsymbol{w}^n$ and $B\eta^n$.

5.1 Estimates in $D(A^{\alpha})$ and $D(B^{\alpha})$, $0 \leq \alpha < 1$

Lemma 5 *Let α be such that $0 \leq \alpha < 1$, under the hypotheses of Theorem 2, the estimates*

$$\|A^{\alpha}\boldsymbol{u}(t) - A^{\alpha}\boldsymbol{v}^n(t)\| \leq \frac{C(\alpha+\varepsilon)}{\lambda_{n+1}^{\varepsilon}}, \quad \|B^{\alpha}\theta(t) - B^{\alpha}\rho^n(t)\| \leq \frac{C(\alpha+\varepsilon)}{\gamma_{n+1}^{\varepsilon}}$$

hold for any $\varepsilon > 0$ such that $0 < \alpha + \varepsilon < 1$.

Proof Observe that the operators A^{ε} and B^{ε} commute with P_n and R_n respectively, for any $\varepsilon \in (0, 1)$. Since A^{ε} and B^{ε} are again positive definite symmetric operators in $\boldsymbol{H}$ and $L^2(\Omega)$, having the eigenvalues λ_n^{ε} and γ_n^{ε} and the eigenfunctions $\boldsymbol{w}^n$ and ω^n, $n = 1, 2, \ldots$ respectively, we can apply Lemma 4 for $f = A^{\alpha}\boldsymbol{u}$, $A^{\star} = A^{\varepsilon}$ and $f = B^{\alpha}\theta$, $A^{\star} = B^{\varepsilon}$ respectively, to obtain the above estimates. □

Theorem 6 *Under the conditions stated in the previous theorem we have*

$$\|A^{\alpha}\boldsymbol{u}(t) - A^{\alpha}\boldsymbol{u}^n(t)\| \leq \frac{C(\alpha+\varepsilon)}{\lambda_{n+1}^{\varepsilon}} + C\left(\frac{1}{\lambda_{n+1}} + \frac{1}{\gamma_{n+1}}\right)^{1/2},$$
$$\|B^{\alpha}\theta(t) - B^{\alpha}\theta^n(t)\| \leq \frac{C(\alpha+\varepsilon)}{\gamma_{n+1}^{\varepsilon}} + C\left(\frac{1}{\lambda_{n+1}} + \frac{1}{\gamma_{n+1}}\right)^{1/2}.$$

Proof Observe that

$$\begin{aligned}\boldsymbol{v}^n(t) &= \exp(-At)P_n\boldsymbol{u}_0 + \int_0^t \exp(-(t-s)A)P_n(\boldsymbol{j}(s)+\theta(s)\boldsymbol{g}(s))ds\\ &\quad - \int_0^t \exp(-(t-s)A)P_n(\boldsymbol{u}(s)\cdot\nabla\boldsymbol{u}(s))ds,\\ \boldsymbol{u}^n(t) &= \exp(-At)P_n\boldsymbol{u}_0 + \int_0^t \exp(-(t-s)A)P_n(\boldsymbol{j}(s)+\theta^n(s)\boldsymbol{g}(s))ds\\ &\quad - \int_0^t \exp(-(t-s)A)P_n(\boldsymbol{u}^n(s)\cdot\nabla\boldsymbol{u}^n(s))ds,\end{aligned}$$

hence,

$$\begin{aligned}\boldsymbol{w}^n(t) &= \int_0^t \exp(-(t-s)A)P_n(\theta(s)-\theta^n(s))\boldsymbol{g}(s)ds\\ &\quad - \int_0^t \exp(-(t-s)A)P_n(\boldsymbol{u}(s)\cdot\nabla\boldsymbol{u}(s)-\boldsymbol{u}^n(s)\cdot\nabla\boldsymbol{u}^n(s))ds.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|A^\alpha\boldsymbol{w}^n(t)\right\| &\le \int_0^t \|A^\alpha\exp(-(t-s)A)\|\|P_n(\theta(s)-\theta^n(s))\boldsymbol{g}(s)\|ds\\ &\quad + \int_0^t \|A^\alpha\exp(-(t-s)A)\|\|P_n(\boldsymbol{u}(s)\cdot\nabla\boldsymbol{u}(s)-\boldsymbol{u}^n(s)\cdot\nabla\boldsymbol{u}^n(s))\|ds.\end{aligned}$$

By (21), we have for all $0<\alpha\le$ e and $t>0$,

$$\|A^\alpha\exp(-(t-s)A)\| \le \frac{1}{(t-s)^\alpha}, \tag{27}$$

and by using that $\|v\|_{L^2}\le C\|v\|_{L^3}\|v\|_{L^6}\le C\|\nabla v\|_{L^2}\|v\|_{H^1}$ for all $v\in H^1$, we obtain that

$$\begin{aligned}\|P_n(\theta(s)-\theta^n(s))\boldsymbol{g}(s)\| &\le \|\theta(s)-\theta^n(s)\|_{L^3}\|\boldsymbol{g}(s)\|_{L^6}\\ &\le \|\nabla\theta(s)-\nabla\theta^n(s)\|\|\boldsymbol{g}(s)\|_{H^1}\\ &\le C\left(\frac{1}{\lambda_{n+1}}+\frac{1}{\gamma_{n+1}}\right)^{1/2}\|\boldsymbol{g}\|_{L^\infty(H^1)}.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_0^t \|A^\alpha\exp(-(t-s)A)P_n(\theta(s)-\theta^n(s))\boldsymbol{g}(s)\|ds\\ &\quad\le C\left(\frac{1}{\lambda_{n+1}}+\frac{1}{\gamma_{n+1}}\right)^{1/2}\|\boldsymbol{g}\|_{L^\infty(H^1)}\int_0^t\frac{1}{(t-s)^\alpha}ds\end{aligned}$$

for $0 < \alpha < 1$ and $t \le T$. Finally,

$$\int_0^t \|A^\alpha \exp(-(t-s)A)P_n(\theta(s)-\theta^n(s))\boldsymbol{g}(s)\|ds \le C\left(\frac{1}{\lambda_{n+1}}+\frac{1}{\gamma_{n+1}}\right)^{1/2}.$$

On the other hand,

$$\begin{aligned}
&\|P_n(\boldsymbol{u}(s)\cdot\nabla\boldsymbol{u}(s)-\boldsymbol{u}^n(s)\cdot\nabla\boldsymbol{u}^n(s))\|\\
&\quad\le \|(\boldsymbol{u}(s)-\boldsymbol{u}^n(s))\cdot\nabla\boldsymbol{u}(s)\| + \|\boldsymbol{u}^n(s)\cdot\nabla(\boldsymbol{u}(s)-\boldsymbol{u}^n(s))\|\\
&\quad\le \|\boldsymbol{u}(s)-\boldsymbol{u}^n(s)\|_{L^3}\|\nabla\boldsymbol{u}(s)\|_{L^6} + \|\boldsymbol{u}(s)\|_{L^\infty}\|\nabla(\boldsymbol{u}(s)-\boldsymbol{u}^n(s))\|\\
&\quad\le C\|\nabla\boldsymbol{u}(s)-\nabla\boldsymbol{u}^n(s)\|\|A\boldsymbol{u}(s)\| + \|A\boldsymbol{u}^n(s)\|\|\nabla\boldsymbol{u}(s)-\nabla\boldsymbol{u}^n(s)\|\\
&\quad\le C\left(\frac{1}{\lambda_{n+1}}+\frac{1}{\gamma_{n+1}}\right)^{1/2}.
\end{aligned}$$

The last inequality is due to (13) and (16). Consequently, for all $0 < \alpha < 1$, we obtain that

$$\begin{aligned}
&\int_0^t \|A^\alpha \exp(-(t-s)A)P_n(\boldsymbol{u}(s)\cdot\nabla\boldsymbol{u}(s)-\boldsymbol{u}^n(s)\cdot\nabla\boldsymbol{u}^n(s))\|ds\\
&\quad\le C\left(\frac{1}{\lambda_{n+1}}+\frac{1}{\gamma_{n+1}}\right)^{1/2}\frac{T^{1-\alpha}}{1-\alpha}.
\end{aligned}$$

Thus, for all $0 < \alpha < 1$, we have that

$$\|A^\alpha \boldsymbol{w}^n(t)\| \le C\left(\frac{1}{\lambda_{n+1}}+\frac{1}{\gamma_{n+1}}\right)^{1/2}, \tag{28}$$

where C depend on $T, \alpha, \partial\Omega, \|\boldsymbol{g}\|_{L^\infty(H^1)}$.

Now, we estimate the error estimate for temperature. We observe that

$$\begin{aligned}
\rho^n(t) &= \exp(-Bt)R_n\theta_0 + \int_0^t \exp(-(t-s)B)(R_n f(s) - R_n(\boldsymbol{u}(s)\cdot\nabla\theta(s))ds,\\
\theta^n(t) &= \exp(-Bt)R_n\theta_0 + \int_0^t \exp(-(t-s)B)(R_n f(s) - R_n(\boldsymbol{u}^n(s)\cdot\nabla\theta^n(s))ds,
\end{aligned}$$

therefore, we obtain that

$$\eta^n(t) = -\int_0^t \exp(-(t-s)B)(R_n(\boldsymbol{u}(s)\cdot\nabla\theta(s)) - R_n(\boldsymbol{u}^n(s)\cdot\nabla\theta^n(s)))ds. \tag{29}$$

Then,

$$\|B^\alpha\eta^n(t)\| \le \int_0^t \|B^\alpha \exp(-(t-s)B)(R_n(\boldsymbol{u}(s)\cdot\nabla\theta(s)) - R_n(\boldsymbol{u}^n(s)\cdot\nabla\theta^n(s))\|ds.$$

We estimate the right hand side of the above inequality as follows:

$$
\begin{aligned}
&\|R_n(\boldsymbol{u}(s)\cdot\nabla\theta(s)-\boldsymbol{u}^n\cdot\nabla\theta^n(s))\|\\
&\quad\le\|(\boldsymbol{u}(s)-\boldsymbol{u}^n(s))\cdot\nabla\theta(s)\|+\|\boldsymbol{u}^n(s)\cdot\nabla(\theta(s)-\theta^n(s))\|\\
&\quad\le\|\boldsymbol{u}(s)-\boldsymbol{u}^n(s)\|_{L^3}\|\nabla\theta(s)\|_{L^6}+\|\boldsymbol{u}^n(s)\|_{L^\infty}\|\nabla(\theta(s)-\theta^n(s))\|\\
&\quad\le C\|\nabla\boldsymbol{u}(s)-\nabla\boldsymbol{u}^n(s)\|+C\|\nabla\theta(s)-\nabla\theta^n(s)\|\\
&\quad\le C\left(\frac{1}{\lambda_{n+1}}+\frac{1}{\gamma_{n+1}}\right)^{1/2},
\end{aligned}
$$

which implies the following estimate for any $0<\alpha<1$

$$
\|B^\alpha\eta^n(t)\|\le C\left(\frac{1}{\lambda_{n+1}}+\frac{1}{\gamma_{n+1}}\right)^{1/2},
$$

where C depend on $\alpha, T, \partial\Omega$.

From Lemma 5 and the estimate (28) we have immediately that

$$
\begin{aligned}
\|A^\alpha\boldsymbol{u}(t)-A^\alpha\boldsymbol{u}^n(t)\|&\le\|A^\alpha\boldsymbol{u}(t)-A^\alpha\boldsymbol{v}^n(t)\|+\|A^\alpha\boldsymbol{w}^n(t)\|\\
&\le\frac{C(\varepsilon)}{\lambda_{n+1}^\varepsilon}+C\left(\frac{1}{\lambda_{n+1}}+\frac{1}{\gamma_{n+1}}\right)^{1/2}.
\end{aligned}
$$

Analogously we prove the result for the temperature. □

5.2 Estimates in $D(A)$ and $D(B)$

Lemma 6 *Under the hypothesis of Theorem 5 the following estimates are true,*

$$
\|A\boldsymbol{u}(t)-A\boldsymbol{v}^n(t)\|\le\frac{C}{\lambda_{n+1}^\varepsilon},\qquad\|B\theta(t)-B\rho^n(t)\|\le\frac{C}{\gamma_{n+1}^\varepsilon}\tag{30}
$$

for $0\le\varepsilon<1/4$.

Proof By using Lemma 4 and Theorem 5, we obtain that

$$
\|A\boldsymbol{u}(t)-A\boldsymbol{v}^n(t)\|=\|A(I-P_n)\boldsymbol{u}(t)\|=\|(I-P_n)A\boldsymbol{u}(t)\|\le\frac{1}{\lambda_{n+1}^\varepsilon}\|A^{1+\varepsilon}\boldsymbol{u}(t)\|.
$$

The proof for the temperature is analogous. □

Lemma 7 *Under the conditions of previous theorem, the following estimates are satisfied*

$$\|A\boldsymbol{w}^n(t)\| \leq \frac{C(\varepsilon)}{\lambda_{n+1}^{\varepsilon}} + C\left(\frac{1}{\lambda_{n+1}} + \frac{1}{\gamma_{n+1}}\right)^{1/2}, \tag{31}$$

$$\|B\eta^n(t)\| \leq \frac{C(\varepsilon)}{\lambda_{n+1}^{\varepsilon}} + \frac{C}{\gamma_{n+1}^{\varepsilon}} + C\left(\frac{1}{\lambda_{n+1}} + \frac{1}{\gamma_{n+1}}\right)^{1/2}. \tag{32}$$

Proof From (5.1), we have for any $\varepsilon \in (0, 1/4)$

$$\begin{aligned}\|A\boldsymbol{w}^n(t)\| \leq & \int_0^t \|A^{1-\varepsilon}\exp(-(t-s)A)\|\|A^{\varepsilon}(P_n[(\theta(s)-\theta^n(s))\boldsymbol{g}(s)])\|ds \\ & + \int_0^t \|A^{1-\varepsilon}\exp(-(t-s)A)\|\|A^{\varepsilon} \\ & (P_n[(\boldsymbol{u}(s)\cdot\nabla\boldsymbol{u}(s) - \boldsymbol{u}^n(s)\cdot\nabla\boldsymbol{u}^n(s))]\|ds. \end{aligned} \tag{33}$$

From some classical interpolation and Sovolev inequalities for three-dimensional domains and the equivalence between the norms $\|\nabla v\|_{L^2}$ and $\|v\|_{H^1}$ we have that

$$\begin{aligned}\|A^{\varepsilon}(P_n[(\theta(s)-\theta^n(s))\boldsymbol{g}(s)])\| & \leq C\|P_n[(\theta(s)-\theta^n(s))\boldsymbol{g}(s)]\| \\ & \leq C\|(\theta(s)-\theta^n(s))\|_{H^1}\|\boldsymbol{g}(s)\|_{\boldsymbol{H}^1} \\ & \leq C\|\nabla(\theta(s)-\theta^n(s))\|\|\boldsymbol{g}\|_{L^{\infty}(H^1)}. \end{aligned} \tag{34}$$

By using Lemma 1 with $\zeta = \eta = \varepsilon$, we obtain

$$\begin{aligned}\|A^{\varepsilon}(P_n[(\boldsymbol{u}(s)\cdot\nabla\boldsymbol{u}(s) - \boldsymbol{u}^n(s)\cdot\nabla\boldsymbol{u}^n(s))]\| & \leq C\|A^{3/4+\varepsilon}(\boldsymbol{u}(s)-\boldsymbol{u}^n(s))\|\|A\boldsymbol{u}(s)\| \\ & + C\|A^{3/4+\varepsilon}\boldsymbol{u}^n(s)\|\|A(\boldsymbol{u}(s)-\boldsymbol{u}^n(s))\|. \end{aligned} \tag{35}$$

From (33), by using (34), (35) and (27), we get that

$$\begin{aligned}\|A\boldsymbol{w}^n(t)\| \leq & \int_0^t \frac{1}{(t-s)^{1-\varepsilon}} C\|\nabla(\theta(s)-\theta^n(s))\|ds \\ & + \int_0^t \frac{1}{(t-s)^{1-\varepsilon}} C(\|A^{3/4+\varepsilon}(\boldsymbol{u}(s)-\boldsymbol{u}^n(s))\| + \|A(\boldsymbol{u}(s)-\boldsymbol{u}^n(s))\|)ds. \end{aligned}$$

From (16) and Theorem 6, we obtain

$$\begin{aligned}\|A\boldsymbol{w}^n(t)\| \leq & C\left(\frac{1}{\lambda_{n+1}} + \frac{1}{\gamma_{n+1}}\right)^{1/2} \int_0^t \frac{1}{(t-s)^{1-\varepsilon}}ds \\ & + \left[\frac{C(\varepsilon)}{\lambda_{n+1}^{\varepsilon}} + \left(\frac{1}{\lambda_{n+1}} + \frac{1}{\gamma_{n+1}}\right)^{1/2}\right] \int_0^t \frac{1}{(t-s)^{1-\varepsilon}}ds \\ & + \int_0^t \frac{1}{(t-s)^{1-\varepsilon}}(\|A\boldsymbol{u}(s) - A\boldsymbol{v}^n(s)\| + \|A\boldsymbol{w}^n(s)\|)ds. \end{aligned}$$

By using Lemma 6, we have that

$$\int_0^t \frac{1}{(t-s)^{1-\varepsilon}} \|A\boldsymbol{u}(s) - A\boldsymbol{v}^n(s)\| ds \le C \frac{1}{\lambda_{n+1}^{\varepsilon}}.$$

Therefore, we obtain

$$\|A\boldsymbol{w}^n(t)\| \le \frac{C(\varepsilon)}{\lambda_{n+1}^{\varepsilon}} + C\left(\frac{1}{\lambda_{n+1}} + \frac{1}{\gamma_{n+1}}\right)^{1/2} + \int_0^t \frac{1}{(t-s)^{1-\varepsilon}} \|A\boldsymbol{w}^n(s)\| ds.$$

By taking Lemma 3 into account, (31) is attained.

In analogous way, we prove the result to the temperature. From (29), by using (22), we have that

$$\begin{aligned}\|B\eta^n(t)\| \le \int_0^t &\|B^{1-\varepsilon}\exp(-(t-s)B)\| \\ &\cdot \|B^{\varepsilon}\left(R_n(\boldsymbol{u}(s)\cdot\nabla\theta(s)) - R_n(\boldsymbol{u}^n(s)\cdot\nabla\theta^n(s))\right)\| ds,\end{aligned}$$

where

$$\begin{aligned}&\|B^{\varepsilon}\left(R_n(\boldsymbol{u}(s)\cdot\nabla\theta(s)) - R_n(\boldsymbol{u}^n(s)\cdot\nabla\theta^n(s))\right)\| \\ &\quad \le C\|A^{3/4+\varepsilon}(\boldsymbol{u}(s) - \boldsymbol{u}^n(s))\|\|B\theta^n(s)\| + C\|A^{3/4+\varepsilon}\boldsymbol{u}^n(s)\|\|B(\theta(s) - \theta^n(s)\| \\ &\quad \le \frac{C(\varepsilon)}{\lambda_{n+1}^{\varepsilon}} + C\left(\frac{1}{\lambda_{n+1}} + \frac{1}{\gamma_{n+1}}\right)^{1/2} + C\|B(\theta(s) - \rho^n(s))\| + C\|B\eta^n(s)\| \\ &\quad \le \frac{C(\varepsilon)}{\lambda_{n+1}^{\varepsilon}} + C\left(\frac{1}{\lambda_{n+1}} + \frac{1}{\gamma_{n+1}}\right)^{1/2} + C\frac{1}{\gamma_{n+1}^{\varepsilon}} + C\|B\eta^n(s)\|.\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}\|B\eta^n(t)\| \le C&\left[\frac{C(\varepsilon)}{\lambda_{n+1}^{\varepsilon}} + \frac{C}{\gamma_{n+1}^{\varepsilon}} + \left(\frac{1}{\lambda_{n+1}} + \frac{1}{\gamma_{n+1}}\right)^{1/2}\right]\int_0^t \frac{1}{(t-s)^{1-\varepsilon}} ds \\ &+ \|B\eta^n(s)\| \int_0^t \frac{1}{(t-s)^{1-\varepsilon}} ds.\end{aligned}$$

Again, Lemma 3 provides (32). □

6 Proof of Theorem 4

By applying Lemmas 6 and 7 to the following splitting

$$\begin{aligned}\|A\boldsymbol{u}(t) - A\boldsymbol{u}^n(t)\| &\le \|A\boldsymbol{u}(t) - A\boldsymbol{v}^n(t)\| + \|A\boldsymbol{w}^n(t)\| \\ \|B\theta(t) - B\theta^n(t)\| &\le \|B\theta(t) - B\rho^n(t)\| + \|B\eta^n(t)\|,\end{aligned}$$

we conclude that

$$\|A\boldsymbol{u}(t) - A\boldsymbol{u}^n(t)\| \leq C\left[\frac{C(\varepsilon)}{\lambda_{n+1}^{\varepsilon}} + \left(\frac{1}{\lambda_{n+1}} + \frac{1}{\gamma_{n+1}}\right)^{1/2}\right]$$

and

$$\|B\theta(t) - B\theta^n(t)\| \leq C\left[\frac{C(\varepsilon)}{\lambda_{n+1}^{\varepsilon}} + \frac{1}{\gamma_{n+1}^{\varepsilon}} + \left(\frac{1}{\gamma_{n+1}} + \frac{1}{\gamma_{n+1}}\right)^{1/2}\right].$$

On the other hand, $\boldsymbol{u}_t - \boldsymbol{u}_t^n$ satisfies

$$\begin{aligned}\boldsymbol{u}_t(t) - \boldsymbol{u}_t^n(t) = &-A(\boldsymbol{u}(t) - \boldsymbol{u}^n(t)) - (P((\boldsymbol{u}(t)\cdot\nabla)\boldsymbol{u}(t)) - P_n((\boldsymbol{u}^n(t)\cdot\nabla\boldsymbol{u}^n(t)))\\ &+P(\boldsymbol{j}(t) - \theta g(t)) - P_n(\boldsymbol{j}(t) - \theta^n g(t)).\end{aligned}$$

Therefore,

$$\begin{aligned}\|\boldsymbol{u}_t(t) - \boldsymbol{u}_t^n(t)\| \leq\ &\|A(\boldsymbol{u}(t) - \boldsymbol{u}^n(t))\| + \|(\boldsymbol{u}(t)\cdot\nabla)(\boldsymbol{u}(t)(t) - \boldsymbol{u}^n(t))\|\\ &+\|((\boldsymbol{u}(t) - \boldsymbol{u}^n(t))\cdot\nabla)\boldsymbol{u}^n(t)\| + \|(\theta^n(t) - \theta(t))\boldsymbol{g}(t)\| \quad (36)\end{aligned}$$

holds. The first term on the right side of (36) is bounded as in (19). For the second and third terms, we use Theorem 2 and (16),

$$\|(\boldsymbol{u}(t)\cdot\nabla)(\boldsymbol{u}(t) - \boldsymbol{u}^n(t))\| \leq \|A\boldsymbol{u}(t)\|\|\nabla\boldsymbol{u}(t) - \nabla\boldsymbol{u}^n(t)\| \leq C\left(\frac{1}{\lambda_{n+1}} + \frac{1}{\gamma_{n+1}}\right)^{1/2},$$

$$\|((\boldsymbol{u}(t) - \boldsymbol{u}^n(t))\cdot\nabla)\boldsymbol{u}^n(t)\| \leq \|A\boldsymbol{u}^n(t)\|\|\nabla\boldsymbol{u}(t) - \nabla\boldsymbol{u}^n(t)\| \leq C\left(\frac{1}{\lambda_{n+1}} + \frac{1}{\gamma_{n+1}}\right)^{1/2}.$$

The last term, also by (16), can be bounded as

$$\|(\theta^n(t) - \theta(t))\boldsymbol{g}(t)\| \leq \|\nabla(\theta^n(t) - \theta(t))\|\|\boldsymbol{g}(t)\|_{H^1} \leq C\left(\frac{1}{\lambda_{n+1}} + \frac{1}{\gamma_{n+1}}\right)^{1/2}.$$

Therefore, we conclude that

$$\left\|\boldsymbol{u}_t(t) - \boldsymbol{u}_t^n(t)\right\| \leq \frac{C(\varepsilon)}{\lambda_{n+1}^{\varepsilon}} + C\left(\frac{1}{\lambda_{n+1}} + \frac{1}{\gamma_{n+1}}\right)^{1/2}.$$

By proceeding in the same way for the temperature we obtain

$$\left\|\theta_t(t) - \theta_t^n(t)\right\| \leq \frac{C(\varepsilon)}{\lambda_{n+1}^{\varepsilon}} + \frac{C}{\gamma_{n+1}^{\varepsilon}} + C\left(\frac{1}{\lambda_{n+1}} + \frac{1}{\gamma_{n+1}}\right)^{1/2}.$$

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